The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function
How to fit a line to a scatterplot?

- some "rule": Least squares? Least absolute deviations?
- visually, by hand (tends to be principal components)
- a statistical criterion: (unbiasedness, efficiency, consistency, etc.)
- a full theory of inference, and for a specific purpose (like causal estimation, prediction, etc.)

(It's a pretty dumb question, don't you think?)
How to fit a line to a scatterplot?

- Some “rule”: Least squares? Least absolute deviations?
- Visually, by hand (tends to be principal components)
- A statistical criterion: (unbiasedness, efficiency, consistency, etc.)
- A full theory of inference, and for a specific purpose (like causal estimation, prediction, etc.)

(It's a pretty dumb question, don't you think?)

The Impossibility of Inference Without Assumptions
How to fit a line to a scatterplot?

• some “rule”: Least squares? Least absolute deviations?
How to fit a line to a scatterplot?

- some “rule”: Least squares? Least absolute deviations?
- visually, by hand (tends to be principal components)
How to fit a line to a scatterplot?

• some “rule”: Least squares? Least absolute deviations?
• visually, by hand (tends to be principal components)
• a statistical criterion: (unbiasedness, efficiency, consistency, etc.)
How to fit a line to a scatterplot?

- **some “rule”:** Least squares? Least absolute deviations?
- **visually, by hand** (tends to be principal components)
- **a statistical criterion:** (unbiasedness, efficiency, consistency, etc.)
- **a full theory of inference, and for a specific purpose** (like causal estimation, prediction, etc.)
How to fit a line to a scatterplot?

- some “rule”: Least squares? Least absolute deviations?
- visually, by hand (tends to be principal components)
- a statistical criterion: (unbiasedness, efficiency, consistency, etc.)
- a full theory of inference, and for a specific purpose (like causal estimation, prediction, etc.)
- (It’s a pretty dumb question, don’t you think?)
Quantities of Interest
Quantities of Interest

- **Summarizing data**: functions of facts you have
Quantities of Interest

- **Summarizing data**: functions of facts you have
- **Inference**: using facts you know to learn facts you don’t know
Quantities of Interest

- **Summarizing data**: functions of facts you have
- **Inference**: using facts you know to learn facts you don’t know
The Problem of Inference

- Probability:
  \( P(y | M) = P(\text{known} | \text{unknown}) \)

- The goal of inverse probability:
  \( P(M | y) = P(\text{unknown} | \text{known}) \)

- A more reasonable, limited goal.
  \[ M = \{ M^*, \theta \}, \text{where } M^* \text{ is assumed} \]
  \( P(\theta | y, M^*) \equiv P(\theta | y) \)
The Problem of Inference

• Probability:

\[ P(y \mid M) = P(\text{known} \mid \text{unknown}) \]
The Problem of Inference

• Probability:

\[ P(y \mid M) = P(\text{known} \mid \text{unknown}) \]

• The goal of inverse probability:

\[ P(M \mid y) = P(\text{unknown} \mid \text{known}) \]
The Problem of Inference

- **Probability:**
  \[ P(y \mid M) = P(\text{known} \mid \text{unknown}) \]

- **The goal of inverse probability:**
  \[ P(M \mid y) = P(\text{unknown} \mid \text{known}) \]

- **A more reasonable, limited goal.** Let \( M = \{M^*, \theta\} \), where \( M^* \) is assumed & \( \theta \) is to be estimated:
  \[ P(\theta \mid y, M^*) \equiv P(\theta \mid y) \]
The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function
Bayes Theorem (as distinct from Bayesian inference):

\[ P(\theta|y) = \frac{P(\theta, y)}{P(y)} \quad \text{[Defn. of conditional probability]} \]

\[ = \frac{P(\theta)P(y|\theta)}{P(y)} \quad \text{[} P(A, B) = P(B)P(A|B)\text{]} \]

\[ = \int P(\theta)P(y|\theta)d\theta \quad \text{[} P(A) = \int P(A, B)dB\text{]} \]

If we knew the right side, we could compute the inverse probability.

Theories of inference arose to interpret this result:

- **Likelihood**
- **Bayesian**

In both, \( P(y|\theta) \) is a traditional probability density.

The two differ on the rest.
Building Theories of Inference

Everything on this page is true; no assumptions

Bayes Theorem (as distinct from Bayesian inference):

\[
P(\theta | y) = \frac{P(\theta, y)}{P(y)} = \frac{P(\theta) P(y | \theta)}{\int P(\theta) P(y | \theta) d\theta}
\]

- If we knew the right side, we could compute the inverse probability.
- Theories of inference arose to interpret this result: Likelihood and Bayesian.
- In both, \( P(y | \theta) \) is a traditional probability density.
- The two differ on the rest.
Building Theories of Inference

Everything on this page is true; no assumptions

- Bayes Theorem (as distinct from Bayesian inference):

\[
P(\theta | y) = \frac{P(\theta, y)}{P(y)} = \frac{P(\theta) P(y | \theta)}{\int P(\theta) P(y | \theta) d\theta}
\]

If we knew the right side, we could compute the inverse probability.

Theories of inference arose to interpret this result: Likelihood and Bayesian.

In both, \( P(y | \theta) \) is a traditional probability density.

The two differ on the rest.
Building Theories of Inference

Everything on this page is true; no assumptions

- Bayes Theorem (as distinct from Bayesian inference):

\[ P(\theta|y) = \frac{P(\theta, y)}{P(y)} \quad [\text{Defn. of conditional probability}] \]
• Bayes Theorem (as distinct from Bayesian inference):  

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)} \quad \text{[Defn. of conditional probability]}
\]

\[
= \frac{P(\theta)P(y|\theta)}{P(y)} \quad \text{[P(A, B) = P(B)P(A|B)]}
\]
Bayes Theorem (as distinct from Bayesian inference):

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)} = \frac{P(\theta)P(y|\theta)}{P(y)} = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)d\theta}
\]

[Defn. of conditional probability]

\[
[P(A, B) = P(B)P(A|B)]
\]

\[
[P(A) = \int P(A, B)dB]
\]
Building Theories of Inference

Everything on this page is true; no assumptions

- Bayes Theorem (as distinct from Bayesian inference):

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)} \quad \text{[Defn. of conditional probability]}
\]

\[
= \frac{P(\theta)P(y|\theta)}{P(y)}
\]

\[
= \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)d\theta}
\]

- If we knew the right side, we could compute the inverse probability.

\[
[P(A, B) = P(B)P(A|B)]
\]

\[
[P(A) = \int P(A, B)dB]
\]
Bayes Theorem (as distinct from Bayesian inference):

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)} = \frac{P(\theta)P(y|\theta)}{P(y)} \quad \text{[Defn. of conditional probability]}
\]

\[
= \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta) d\theta} \quad \text{[}P(A) = \int P(A, B) dB\text{]} \]

• If we knew the right side, we could compute the inverse probability.

• Theories of inference arose to *interpret* this result:
  
  Likelihood and Bayesian
Building Theories of Inference

Everything on this page is true; no assumptions

- Bayes Theorem (as distinct from Bayesian inference):

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)}
\]

[Dfn. of conditional probability]

\[
= \frac{P(\theta)P(y|\theta)}{P(y)}
\]

[\(P(A, B) = P(B)P(A|B)\)]

\[
= \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)d\theta}
\]

[\(P(A) = \int P(A, B)dB\)]

- If we knew the right side, we could compute the inverse probability.

- Theories of inference arose to *interpret* this result:
  Likelihood and Bayesian

- In both, \(P(y|\theta)\) is a traditional probability density
Bayes Theorem (as distinct from Bayesian inference):

\[ P(\theta|y) = \frac{P(\theta, y)}{P(y)} \]

\[ = \frac{P(\theta)P(y|\theta)}{P(y)} \]

\[ = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)d\theta} \]

If we knew the right side, we could compute the inverse probability.

Theories of inference arose to interpret this result:

Likelihood and Bayesian

In both, \( P(y|\theta) \) is a traditional probability density

The two differ on the rest
Interpretation 1: The Likelihood Theory of Inference

- R.A. Fisher's idea
- $\theta$ is fixed and $y$ is random

Let:

$$k(y) \equiv \int P(\theta)P(y|\theta)\,d\theta$$

$$\Rightarrow P(\theta|y) = \frac{P(\theta)}{k(y)} P(y|\theta)$$

- Define $k(y)$ as an unknown function of $y$ with $\theta$ fixed at its true value

The likelihood theory of inference has four axioms: the 3 probability axioms plus the likelihood axiom (neither true nor false):

$$L(\theta|y) \equiv k(y)P(y|\theta) \propto P(y|\theta)$$
Interpretation 1: The Likelihood Theory of Inference

- R.A. Fisher’s idea
Interpretation 1: The Likelihood Theory of Inference

- R.A. Fisher’s idea
- $\theta$ is fixed and $y$ is random
Interpretation 1: The Likelihood Theory of Inference

- R.A. Fisher’s idea
- $\theta$ is fixed and $y$ is random
- Let:

$$k(y) \equiv \frac{P(\theta)}{\int P(\theta)P(\theta|\theta) d\theta} \implies P(\theta|y) = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta) d\theta} = k(y)P(y|\theta)$$
Interpretation 1: The Likelihood Theory of Inference

- R.A. Fisher’s idea
- $\theta$ is fixed and $y$ is random
- Let:

$$k(y) \equiv \frac{P(\theta)}{\int P(\theta)P(y|\theta)d\theta} \quad \implies \quad P(\theta|y) = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)d\theta} = k(y)P(y|\theta)$$

- Define $k(y)$ as an unknown function of $y$ with $\theta$ fixed at its true value
Interpretation 1: The Likelihood Theory of Inference

- R.A. Fisher’s idea
- $\theta$ is fixed and $y$ is random
- Let:

\[ k(y) \equiv \frac{P(\theta)}{\int P(\theta)P(y|\theta)d\theta} \implies P(\theta|y) = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)d\theta} = k(y)P(y|\theta) \]

- Define $k(y)$ as an unknown function of $y$ with $\theta$ fixed at its true value
- The likelihood theory of inference has four axioms: the 3 probability axioms plus the likelihood axiom (neither true nor false):
Interpretation 1: The Likelihood Theory of Inference

- R.A. Fisher’s idea
- \( \theta \) is fixed and \( y \) is random
- Let:

\[
k(y) \equiv \frac{P(\theta)}{\int P(\theta)P(y|\theta)\,d\theta} \quad \Rightarrow \quad P(\theta|y) = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)\,d\theta} = k(y)P(y|\theta)
\]

- Define \( k(y) \) as an unknown function of \( y \) with \( \theta \) fixed at its true value
- The likelihood theory of inference has four axioms: the 3 probability axioms plus the likelihood axiom (neither true nor false):

\[
L(\theta|y) \equiv k(y)P(y|\theta)
\]
Interpretation 1: The Likelihood Theory of Inference

• R.A. Fisher’s idea
• $\theta$ is fixed and $y$ is random
• Let:

$$k(y) \equiv \frac{P(\theta)}{\int P(\theta)P(y|\theta) d\theta} \implies P(\theta|y) = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta) d\theta} = k(y)P(y|\theta)$$

• Define $k(y)$ as an unknown function of $y$ with $\theta$ fixed at its true value

• The likelihood theory of inference has four axioms: the 3 probability axioms plus the likelihood axiom (neither true nor false):

$$L(\theta|y) \equiv k(y)P(y|\theta)$$

$$\propto P(y|\theta)$$
Interpretation 1: The Likelihood Theory of Inference

- $L(\theta|y)$ is a function: for $y$ fixed at the observed values, it gives the “likelihood” of any value of $\theta$ you might want to try.
Interpretation 1: The Likelihood Theory of Inference

- \( L(\theta|y) \) is a function: for \( y \) fixed at the observed values, it gives the “likelihood” of any value of \( \theta \) you might want to try.
- Likelihood: a relative measure of uncertainty, changing with the data.

The likelihood principle: the data \( y \) only affect inferences through the likelihood function, \( L(\theta|y) = k(y)P(y|\theta) \).
Interpretation 1: The Likelihood Theory of Inference

- $L(\theta|y)$ is a function: for $y$ fixed at the observed values, it gives the “likelihood” of any value of $\theta$ you might want to try.
- Likelihood: a relative measure of uncertainty, changing with the data.
- Comparing value of $L(\theta|y)$ for different $\theta$ values:
Interpretation 1: The Likelihood Theory of Inference

- $L(\theta|y)$ is a function: for $y$ fixed at the observed values, it gives the “likelihood” of any value of $\theta$ you might want to try.
- Likelihood: a relative measure of uncertainty, changing with the data.
- Comparing value of $L(\theta|y)$ for different $\theta$ values:
  - within a data set: meaningful
Interpretation 1: The Likelihood Theory of Inference

- $L(\theta|y)$ is a function: for $y$ fixed at the observed values, it gives the “likelihood” of any value of $\theta$ you might want to try
- Likelihood: a relative measure of uncertainty, changing with the data
- Comparing value of $L(\theta|y)$ for different $\theta$ values:
  - within a data set: meaningful
  - across data sets: meaningless
Interpretation 1: The Likelihood Theory of Inference

- **$L(\theta|y)$ is a function:** for $y$ fixed at the observed values, it gives the “likelihood” of any value of $\theta$ you might want to try.
- Likelihood: a *relative measure of uncertainty*, changing with the data.
- **Comparing value of $L(\theta|y)$ for different $\theta$ values:**
  - within a data set: meaningful
  - across data sets: meaningless
  - You also can’t compare $R^2$ values across equations with different dependent variables.
Interpretation 1: The Likelihood Theory of Inference

• \( L(\theta|y) \) is a function: for \( y \) fixed at the observed values, it gives the “likelihood” of any value of \( \theta \) you might want to try

• Likelihood: a relative measure of uncertainty, changing with the data

• Comparing value of \( L(\theta|y) \) for different \( \theta \) values:
  • within a data set: meaningful
  • across data sets: meaningless
  • You also can’t compare \( R^2 \) values across equations with different dependent variables

• The likelihood principle: the data \( y \) only affect inferences through the likelihood function, \( L(\theta|y) = k(y)P(y|\theta) \)
Visualizing the Likelihood

• For algebraic simplicity and numerical stability, we use the log-likelihood (the shape changes; the max is unchanged)

• If \( \theta \) has one element, we can plot:

• Summary Estimator: The likelihood curve. (Likelihood principle: we can now discard the data—if the model is correct!)

• One-point summary: at the maximum is the "MLE"

• Uncertainty of the MLE: curvature at the maximum
Visualizing the Likelihood

• For algebraic simplicity and numerical stability, we use the log-likelihood (the shape changes; the max is unchanged)
Visualizing the Likelihood

- For algebraic simplicity and numerical stability, we use the log-likelihood (the shape changes; the max is unchanged)
- If $\theta$ has one element, we can plot:
Visualizing the Likelihood

- For algebraic simplicity and numerical stability, we use the **log-likelihood** (the shape changes; the max is unchanged)
- If $\theta$ has one element, we can plot:

*Summary Estimator*: The likelihood curve. *(Likelihood principle: we can now discard the data—if the model is correct!)*
Visualizing the Likelihood

- For algebraic simplicity and numerical stability, we use the log-likelihood (the shape changes; the max is unchanged)
- If $\theta$ has one element, we can plot:

  ![Likelihood Curve Diagram]

  - **Summary Estimator**: The likelihood curve. (Likelihood principle: we can now discard the data—if the model is correct!)
  - **One-point summary**: at the maximum is the “MLE”
Visualizing the Likelihood

- For algebraic simplicity and numerical stability, we use the **log-likelihood** (the shape changes; the max is unchanged)
- If $\theta$ has one element, we can plot:

- **Summary Estimator**: The likelihood curve. (**Likelihood principle**: we can now discard the data—if the model is correct!)
- **One-point summary**: at the maximum is the “MLE”
- **Uncertainty of the MLE**: curvature at the maximum
Interpretation 2: The Bayesian Theory of Inference

• Rev. Thomas Bayes' unpublished idea, and later rediscovered.

Recall:

\[ P(\theta | y) = \frac{P(\theta, y)}{P(y)} \]  
[Defn. of conditional probability]

\[ = P(\theta) P(y | \theta) P(y) \]  
\[ \propto P(\theta) P(y | \theta) \int P(\theta) P(y | \theta) d\theta \]  
\[ P(A) = \int P(A|B) dB \]  

• \( P(\theta | y) \) the posterior density
• \( P(y | \theta) \) the traditional probability (\( \propto \) likelihood)
• \( P(y) \) a constant, easily computed
• \( P(\theta) \), the prior density — the way Bayes differs from likelihood
Interpretation 2: The Bayesian Theory of Inference

• Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
Interpretation 2: The Bayesian Theory of Inference

• Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
• Recall:

\[ P(\theta|y) = \frac{P(\theta, y)}{P(y)} \]
\[ = P(\theta) P(y|\theta) \]
\[ \propto P(\theta) P(y|\theta) \]

the posterior density

\[ P(y|\theta) \]
the traditional probability (\( \propto \) likelihood)

\[ P(y) \]
a constant, easily computed

\[ P(\theta) \]
the prior density

— the way Bayes differs from likelihood
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[ P(\theta | y) = \frac{P(\theta, y)}{P(y)} \]  

[Defn. of conditional probability]
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)} = \frac{P(\theta)P(y|\theta)}{P(y)}
\]

[Defn. of conditional probability]

\[
P(AB) = P(B)P(A|B)
\]

- The posterior density:
- The traditional probability ($\propto$ likelihood)
- A constant, easily computed
- The prior density — the way Bayes differs from likelihood
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)}
\]

[Defn. of conditional probability]

\[
= \frac{P(\theta)P(y|\theta)}{P(y)}
\]

[\(P(AB) = P(B)P(A|B)\)]

\[
= \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta) \, d\theta}
\]

[\(P(A) = \int P(AB) \, dB\)]
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[ P(\theta|y) = \frac{P(\theta, y)}{P(y)} \]

\[ = \frac{P(\theta)P(y|\theta)}{P(y)} \]

\[ = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta) d\theta} \]

\[ \propto P(\theta)P(y|\theta) \]

[Defn. of conditional probability]

\[ P(\theta|y) \text{ the posterior density} \]

\[ P(y|\theta) \text{ the traditional probability (} \propto \text{ likelihood}) \]

\[ P(y) \text{ a constant, easily computed} \]

\[ P(\theta) \text{, the prior density} \]

— the way Bayes differs from likelihood
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)}
\]

\[
= \frac{P(\theta)P(y|\theta)}{P(y)}
\]

\[
= \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)\,d\theta}
\]

\[
\propto P(\theta)P(y|\theta)
\]

- \(P(\theta|y)\) the posterior density

[Defn. of conditional probability]

\[P(AB) = P(B)P(A|B)\]

\[P(A) = \int P(AB)\,dB\]
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[
P(\theta|y) = \frac{P(\theta, y)}{P(y)}
\]

\[
= \frac{P(\theta)P(y|\theta)}{P(y)}
\]

\[
= \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)\,d\theta}
\]

\[
\propto P(\theta)P(y|\theta)
\]

- \(P(\theta|y)\) the posterior density
- \(P(y|\theta)\) the traditional probability \((\propto\) likelihood\)

\[
P(A) = \int P(AB)\,dB
\]

\[
P(AB) = P(B)P(A|B)
\]
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[ P(\theta|y) = \frac{P(\theta, y)}{P(y)} = \frac{P(\theta)P(y|\theta)}{P(y)} = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)\,d\theta} \propto P(\theta)P(y|\theta) \]

- \( P(\theta|y) \) the posterior density
- \( P(y|\theta) \) the traditional probability (\( \propto \) likelihood)
- \( P(y) \) a constant, easily computed
Interpretation 2: The Bayesian Theory of Inference

- Rev. Thomas Bayes’ unpublished idea, and later rediscovered.
- Recall:

\[ P(\theta|y) = \frac{P(\theta, y)}{P(y)} \]
\[ = \frac{P(\theta)P(y|\theta)}{P(y)} \]
\[ = \frac{P(\theta)P(y|\theta)}{\int P(\theta)P(y|\theta)d\theta} \]
\[ \propto P(\theta)P(y|\theta) \]

- \( P(\theta|y) \) the posterior density
- \( P(y|\theta) \) the traditional probability (\( \propto \) likelihood)
- \( P(y) \) a constant, easily computed
- \( P(\theta) \), the prior density — the way Bayes differs from likelihood
What is the prior density, $P(\theta)$?
What is the prior density, $P(\theta)$?

• A probability density that represents all prior evidence about $\theta$
What is the prior density, $P(\theta)$?

- A probability density that represents all prior evidence about $\theta$
- An opportunity: a way of getting other information outside the data set into the model and estimator
What is the prior density, $P(\theta)$?

- A probability density that represents all prior evidence about $\theta$
- An opportunity: a way of getting other information outside the data set into the model and estimator
- An annoyance: the “other information” is required
What is the prior density, \( P(\theta) \)?

- A probability density that represents all prior evidence about \( \theta \)
- An opportunity: a way of getting other information outside the data set into the model and estimator
- An annoyance: the “other information” is required
- A philosophical assumption that nonsample information:
What is the prior density, $P(\theta)$?

- **A probability density** that represents all prior evidence about $\theta$
- **An opportunity**: a way of getting other information outside the data set into the model and estimator
- **An annoyance**: the “other information” is required
- **A philosophical assumption** that nonsample information:
  - should matter — as it always does
What is the prior density, $P(\theta)$?

- **A probability density** that represents all prior evidence about $\theta$
- **An opportunity**: a way of getting other information outside the data set into the model and estimator
- **An annoyance**: the “other information” is required
- **A philosophical assumption** that nonsample information:
  - should matter — as it always does
  - should be formalized and included in all inferences — which is more debatable
What is the prior density, $P(\theta)$?

- A probability density that represents all prior evidence about $\theta$
- An opportunity: a way of getting other information outside the data set into the model and estimator
- An annoyance: the “other information” is required
- A philosophical assumption that nonsample information:
  - should matter — as it always does
  - should be formalized and included in all inferences — which is more debatable
- Quiz: Example of nonsample information you want included
What is the prior density, $P(\theta)$?

- A probability density that represents all prior evidence about $\theta$
- An opportunity: a way of getting other information outside the data set into the model and estimator
- An annoyance: the “other information” is required
- A philosophical assumption that nonsample information:
  - should matter — as it always does
  - should be formalized and included in all inferences — which is more debatable
- Quiz: Example of nonsample information you want included
- Quiz 2: Example of nonsample information you’re skeptical of including
Principles of Bayesian analysis

1. All unknown quantities ($\theta$, $Y$) are treated as random variables and have a joint probability distribution.
2. All known quantities ($y$) are treated as fixed.
3. If we have observed variable $B$ and unobserved variable $A$, then we are usually interested in the conditional distribution of $A$, given $B$:
   \[ P(A \mid B) = \frac{P(A, B)}{P(B)} \]
4. If variables $A$ and $B$ are both unknown, then the distribution of $A$ alone is:
   \[ P(A) = \int P(A, B) dB = \int P(A \mid B) P(B) dB \]
1. All unknown quantities ($\theta$, $Y$) are treated as random variables and have a joint probability distribution.
Principles of Bayesian analysis

1. All unknown quantities ($\theta, Y$) are treated as random variables and have a joint probability distribution.
2. All known quantities ($y$) are treated as fixed.

3. If we have observed variable $B$ and unobserved variable $A$, then we are usually interested in the conditional distribution of $A$, given $B$:
   \[ P(A \mid B) = \frac{P(A, B)}{P(B)} \]

4. If variables $A$ and $B$ are both unknown, then the distribution of $A$ alone is:
   \[ P(A) = \int P(A, B) dB = \int P(A \mid B) P(B) dB \]
1. All unknown quantities ($\theta, Y$) are treated as random variables and have a joint probability distribution.

2. All known quantities ($y$) are treated as fixed.

3. If we have observed variable $B$ and unobserved variable $A$, then we are usually interested in the conditional distribution of $A$, given $B$: 

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$
Principles of Bayesian analysis

1. All unknown quantities ($\theta, Y$) are treated as random variables and have a joint probability distribution.
2. All known quantities ($y$) are treated as fixed.
3. If we have observed variable $B$ and unobserved variable $A$, then we are usually interested in the conditional distribution of $A$, given $B$:
   \[
P(A \mid B) = \frac{P(A, B)}{P(B)}\]
4. If variables $A$ and $B$ are both unknown, then the distribution of $A$ alone is $P(A) = \int P(A, B)dB = \int P(A \mid B)P(B)dB$. 
The posterior density, $P(\theta|y)$

Unlike $L$, it's a real probability density, from which we can derive probabilistic statements (via integration).

To compare across applications or data sets, you may need different priors. So, the posterior is relative, just like likelihood.

Bayesian inference obeys the likelihood principle: the data set only affects inferences through the likelihood function.

If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$. 
The posterior density, $P(\theta|y)$

- Like $L$, it’s a summary estimator
The posterior density, $P(\theta|y)$

- Like $L$, it’s a **summary estimator**
- Unlike $L$, it’s a **real probability density**, from which we can derive probabilistic statements (via integration)
The posterior density, $P(\theta|y)$

- Like $L$, it’s a **summary estimator**
- Unlike $L$, it’s a **real probability density**, from which we can derive probabilistic statements (via integration)
- To compare across applications or data sets, you may need different priors. So, the **posterior is relative**, just like likelihood.
The posterior density, $P(\theta|y)$

- Like $L$, it’s a summary estimator.
- Unlike $L$, it’s a real probability density, from which we can derive probabilistic statements (via integration).
- To compare across applications or data sets, you may need different priors. So, the posterior is relative, just like likelihood.
- Bayesian inference obeys the likelihood principle: the data set only affects inferences through the likelihood function.
The posterior density, $P(\theta|y)$

- Like $L$, it’s a **summary estimator**
- Unlike $L$, it’s a **real probability density**, from which we can derive probabilistic statements (via integration)
- To compare across applications or data sets, you may need different priors. So, the **posterior is relative**, just like likelihood.
- **Bayesian inference** obeys the **likelihood principle**: the data set only affects inferences through the likelihood function
- If $P(\theta) = 1$, i.e., is uniform in the relevant region, then $L(\theta|y) = P(\theta|y)$. 

Three Theories of Inference: Overview
How to think about Bayes v. Likelihood

Summary:
- Likelihood is simpler; start there
- Bayes opens up more possibilities; use if needed

Philosophical differences from likelihood:
- Huge

Practical differences:
- Minor, unless the prior matters

Example where prior matters:
- Demographic forecasting model

Bayesians are happier people:
- If \( P(\theta) \) is diffuse, differences from likelihood are minor, but numerical stability (and "identification") improves \( \Rightarrow \) your programs will run better!

Advantages of Bayes:
- More information \( \Rightarrow \) more efficiency; MCMC algorithms are easier

Few fights now between Bayesians and likelihoodists
How to think about Bayes v. Likelihood

• Summary:

Likelihood is simpler; start there
Bayes opens up more possibilities; use if needed
Philosophical differences from likelihood: Huge
Practical differences: Minor, unless the prior matters
Example where prior matters: demographic forecasting model
Bayesians are happier people: If \( P(\theta) \) is diffuse, differences from likelihood are minor, but numerical stability (and “identification”) improves \( \Rightarrow \) your programs will run better!
Advantages of Bayes: more information \( \Rightarrow \) more efficiency; MCMC algorithms are easier
Few fights now between Bayesians and likelihoodists
How to think about Bayes v. Likelihood

• **Summary:**
  • Likelihood is simpler; start there
How to think about Bayes v. Likelihood

• **Summary:**
  - Likelihood is simpler; start there
  - Bayes opens up more possibilities; use if needed
How to think about Bayes v. Likelihood

- **Summary:**
  - Likelihood is simpler; start there
  - Bayes opens up more possibilities; use if needed
- **Philosophical differences from likelihood:** Huge

Practical differences: Minor, unless the prior matters

Example where prior matters: demographic forecasting model

Bayesians are happier people: If \( P(\theta) \) is diffuse, differences from likelihood are minor, but numerical stability (and "identification") improves \( \Rightarrow \) your programs will run better!

Advantages of Bayes: more information \( \Rightarrow \) more efficiency; MCMC algorithms are easier

Few fights now between Bayesians and likelihoodists
How to think about Bayes v. Likelihood

• **Summary:**
  - Likelihood is simpler; start there
  - Bayes opens up more possibilities; use if needed

• **Philosophical differences from likelihood:** Huge

• **Practical differences:** Minor, unless the prior matters
How to think about Bayes v. Likelihood

• **Summary:**
  • Likelihood is simpler; start there
  • Bayes opens up more possibilities; use if needed

• **Philosophical differences from likelihood:** Huge

• **Practical differences:** Minor, unless the prior matters

• **Example where prior matters:** demographic forecasting model
How to think about Bayes v. Likelihood

- **Summary:**
  - Likelihood is simpler; start there
  - Bayes opens up more possibilities; use if needed
- **Philosophical differences from likelihood:** Huge
- **Practical differences:** Minor, unless the prior matters
- **Example where prior matters:** demographic forecasting model
- **Bayesians are happier people:** If $P(\theta)$ is diffuse, differences from likelihood are minor, but numerical stability (and “identification”) improves $\Rightarrow$ your programs will run better!
How to think about Bayes v. Likelihood

• **Summary:**
  - Likelihood is simpler; start there
  - Bayes opens up more possibilities; use if needed

• **Philosophical differences from likelihood:** Huge

• **Practical differences:** Minor, unless the prior matters

• **Example where prior matters:** demographic forecasting model

• **Bayesians are happier people:** If \( P(\theta) \) is *diffuse*, differences from likelihood are minor, but numerical stability (and “identification”) improves \( \Rightarrow \) your programs will run better!

• **Advantages of Bayes:** more information \( \Rightarrow \) more efficiency; MCMC algorithms are easier
How to think about Bayes v. Likelihood

- **Summary:**
  - Likelihood is simpler; start there
  - Bayes opens up more possibilities; use if needed
- **Philosophical differences from likelihood:** Huge
- **Practical differences:** Minor, unless the prior matters
- **Example where prior matters:** demographic forecasting model
- **Bayesians are happier people:** If \( P(\theta) \) is *diffuse*, differences from likelihood are minor, but numerical stability (and “identification”) improves \( \Leftrightarrow \) your programs will run better!
- **Advantages of Bayes:** more information \( \Leftrightarrow \) more efficiency; MCMC algorithms are easier
- **Few fights now** between Bayesians and likelihoodists
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the {Bayesians, Likelihoodists}

2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$

3. All tests are "under" (i.e., assuming) $H_0$. For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?

- $H_0$: $\beta = 0$ vs. $H_1$: $\beta > 0$

- Choose Type I error, probability of deciding $H_1$ is right when $H_0$ is really true: say $\alpha = 0.05$

- (Type II error, the power to detect $H_1$ if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing $\alpha$.)

- Assume $n$ is large enough for the CLT to kick in

- Then $b|(\beta = 0) \sim N(0, \sigma_b^2)$

- or $(T_S)\beta|(\beta = 0) \equiv b - \hat{\beta} \sigma_b \equiv b \sim N(0, 1)$.
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the {Bayesians, Likelihoodists}
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the \{Bayesians, Likelihoodists\}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the {Bayesians, Likelihoodists}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the {Bayesians, Likelihoodists}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$

For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the \{Bayesians, Likelihoodists\}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$

For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?

- $H_0$: $\beta = 0$ vs. $H_1$: $\beta > 0$
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the \{Bayesians, Likelihoodists\}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$

For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?

- $H_0: \beta = 0$ vs. $H_1: \beta > 0$
- Choose Type I error, probability of deciding $H_1$ is right when $H_0$ is really true: say $\alpha = 0.05$
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the \{Bayesians, Likelihoodists\}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$

For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?

- $H_0$: $\beta = 0$ vs. $H_1$: $\beta > 0$
- Choose Type I error, probability of deciding $H_1$ is right when $H_0$ is really true: say $\alpha = 0.05$
- (Type II error, the power to detect $H_1$ if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing $\alpha$.)
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the {Bayesians, Likelihoodists}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$

For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?

- $H_0$: $\beta = 0$ vs. $H_1$: $\beta > 0$
- Choose Type I error, probability of deciding $H_1$ is right when $H_0$ is really true: say $\alpha = 0.05$
- (Type II error, the power to detect $H_1$ if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing $\alpha$.)
- Assume $n$ is large enough for the CLT to kick in
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the {Bayesians, Likelihoodists}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$

For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?

- $H_0: \beta = 0$ vs. $H_1: \beta > 0$
- Choose Type I error, probability of deciding $H_1$ is right when $H_0$ is really true: say $\alpha = 0.05$
- (Type II error, the power to detect $H_1$ if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing $\alpha$.)
- Assume $n$ is large enough for the CLT to kick in
- Then $b|(\beta = 0) \sim N(0, \sigma_b^2)$
A 3rd Theory: Neyman-Pearson Hypothesis Testing

1. Fights between these folks and the {Bayesians, Likelihoodists}
2. Strict but arbitrary distinction: null $H_0$ vs alternative $H_1$
3. All tests are “under” (i.e., assuming) $H_0$

For example, is $\beta = 0$ in $E(Y) = \beta_0 + \beta X$?

- $H_0: \beta = 0$ vs. $H_1: \beta > 0$
- Choose Type I error, probability of deciding $H_1$ is right when $H_0$ is really true: say $\alpha = 0.05$
- (Type II error, the power to detect $H_1$ if it is true, is a consequence of choosing an estimator, not an ex ante decision like choosing $\alpha$.)
- Assume $n$ is large enough for the CLT to kick in
- Then $b|\beta = 0 \sim N(0, \sigma_b^2)$
- or

\[
(TS)\beta|\beta = 0 \equiv \frac{b - \beta}{\hat{\sigma}_b} \equiv \frac{b}{\hat{\sigma}_b} \sim N(0, 1).
\]
Neyman-Pearson Hypothesis Testing

• Derive critical value, $C_V$, e.g., the right tail:

$$\int_\infty^{C_V} N(b|0, \sigma^2_b)db = \alpha$$

• In educational psychology and some other fields: write your prospectus, plan your experiment, report the $C_V$, and write the concluding chapter:

\[
\text{Decision} = \begin{cases} 
\beta > 0 & (I was right) \text{ if } (T_S) > (C_V) \\
\beta = 0 & (I was wrong) \text{ if } (T_S) \leq (C_V) 
\end{cases}
\]

• Then collect your data. You may not revise your hypothesis or chapter.

Once discredited; making a comeback through the preregistration movement.
Neyman-Pearson Hypothesis Testing

- **Derive critical value**, $CV$, e.g., the right tail:

\[ \int_{(CV)}^{\infty} N(b|0, \sigma^2_b) \, db = \alpha \]
Neyman-Pearson Hypothesis Testing

- **Derive critical value**, $CV$, e.g., the right tail:

$$\int_{CV}^{\infty} N(b|0, \sigma_b^2) db = \alpha$$

- **In educational psychology and some other fields:** write your prospectus, plan your experiment, report the $CV$, and write the concluding chapter:
Neyman-Pearson Hypothesis Testing

- Derive critical value, $CV$, e.g., the right tail:

$$\int_{(CV)}^{\infty} N(b|0, \sigma_b^2) db = \alpha$$

- In educational psychology and some other fields: write your prospectus, plan your experiment, report the $CV$, and write the concluding chapter:

Decision =
Neyman-Pearson Hypothesis Testing

- **Derive critical value**, $CV$, e.g., the right tail:

  $$\int_{(CV)}^{\infty} N(b|0, \sigma_b^2) db = \alpha$$

- **In educational psychology and some other fields:** write your prospectus, plan your experiment, report the $CV$, and write the concluding chapter:

  \[
  \text{Decision} = \begin{cases} 
  \beta > 0 \text{ (I was right)} & \text{if } (TS) > (CV) \\
  \beta = 0 \text{ (I was wrong)} & \text{if } (TS) \leq (CV)
  \end{cases}
  \]
Neyman-Pearson Hypothesis Testing

- Derive critical value, $CV$, e.g., the right tail:

$$\int_{(CV)}^{\infty} N(b|0, \sigma_b^2) \, db = \alpha$$

- In educational psychology and some other fields: write your prospectus, plan your experiment, report the $CV$, and write the concluding chapter:

$$\text{Decision} = \begin{cases} 
\beta > 0 \text{ (I was right)} & \text{if } (TS) > (CV) \\
\beta = 0 \text{ (I was wrong)} & \text{if } (TS) \leq (CV)
\end{cases}$$

- Then collect your data. You may not revise your hypothesis or chapter.
Neyman-Pearson Hypothesis Testing

• Derive critical value, $CV$, e.g., the right tail:

$$\int_{(CV)}^{\infty} N(b|0, \sigma^2_{b}) \, db = \alpha$$

• In educational psychology and some other fields: write your prospectus, plan your experiment, report the $CV$, and write the concluding chapter:

$$\text{Decision} = \begin{cases} 
\beta > 0 \text{ (I was right)} & \text{if } (TS) > (CV) \\
\beta = 0 \text{ (I was wrong)} & \text{if } (TS) \leq (CV)
\end{cases}$$

• Then collect your data. You may not revise your hypothesis or chapter

• Once discredited; making a comeback through the preregistration movement
Neyman-Pearson Hypothesis Testing

In this example, \((T_S) < (C_V) \Rightarrow \) conclude \(\beta = 0\).

Decision will be wrong 5% of the time.

Quiz: What is the probability it's right this time?

Quiz 2: What happens when \(n\) is large (or under your control)?

Relaxed approach, use \(p\)-values:
The probability under the null of getting a value as or more extreme than the value we got — the area to the right of the realized value of \((T_S)\).

Star-gazing is often silly; where's the QOI?

\(\Rightarrow\) Can use likelihood: to compute tests and \(p\)-values.

Three Theories of Inference: Overview
Neyman-Pearson Hypothesis Testing

In this example, $(T_S) < (CV) \Rightarrow$ conclude $\beta = 0$.

Decision will be wrong 5% of the time

Quiz: What is the probability it's right this time?

Quiz 2: What happens when $n$ is large (or under your control)?

Relaxed approach, use $p$-values: The probability under the null of getting a value as or more extreme than the value we got — the area to the right of the realized value of $(T_S)$.

Star-gazing is often silly; where's the QOI?

$\Rightarrow$ Can use likelihood: to compute tests and $p$-values.

Three Theories of Inference: Overview
Neyman-Pearson Hypothesis Testing

- In this example, $(TS) < (CV) \implies$ conclude $\beta = 0$. 

Three Theories of Inference: Overview
Neyman-Pearson Hypothesis Testing

- In this example, \((TS) < (CV) \Rightarrow\) conclude \(\beta = 0\).
- Decision will be wrong 5% of the time.
Neyman-Pearson Hypothesis Testing

- In this example, $(TS) < (CV) \implies$ conclude $\beta = 0$.
- Decision will be wrong 5% of the time
- Quiz: What is the probability it’s right this time?

Star-gazing is often silly; where’s the QOI?

→ Can use likelihood: to compute tests and $p$-values.

Relaxed approach, use $p$-values: The probability under the null of getting a value as or more extreme than the value we got — the area to the right of the realized value of $(TS)$. 

Three Theories of Inference: Overview
Neyman-Pearson Hypothesis Testing

• In this example, \((TS) < (CV) \Rightarrow\) conclude \(\beta = 0\).
• Decision will be wrong 5\% of the time
• Quiz: What is the probability it’s right this time?
• Quiz 2: What happens when \(n\) is large (or under your control)?
Neyman-Pearson Hypothesis Testing

- In this example, \((TS) < (CV) \Rightarrow\) conclude \(\beta = 0\).
- Decision will be wrong 5% of the time
- Quiz: What is the probability it’s right this time?
- Quiz 2: What happens when \(n\) is large (or under your control)?
- Relaxed approach, use \(p\)-values:
Neyman-Pearson Hypothesis Testing

- In this example, \((TS) < (CV) \Rightarrow \text{conclude } \beta = 0\).
- Decision will be wrong 5% of the time.
- Quiz: What is the probability it’s right this time?
- Quiz 2: What happens when \(n\) is large (or under your control)?
- Relaxed approach, use \(p\)-values: The probability under the null of getting a value as or more extreme than the value we got — the area to the right of the realized value of \((TS)\).
Neyman-Pearson Hypothesis Testing

- In this example, $(TS) < (CV) \Rightarrow$ conclude $\beta = 0$.
- Decision will be wrong 5% of the time
- Quiz: What is the probability it’s right this time?
- Quiz 2: What happens when $n$ is large (or under your control)?
- Relaxed approach, use $p$-values: The probability under the null of getting a value as or more extreme than the value we got — the area to the right of the realized value of $(TS)$.
- Star-gazing is often silly; where’s the QOI?
In this example, \((TS) < (CV) \Rightarrow \text{conclude } \beta = 0\).

Decision will be wrong 5% of the time.

Quiz: What is the probability it’s right this time?

Quiz 2: What happens when \(n\) is large (or under your control)?

Relaxed approach, use \(p\)-values: The probability under the null of getting a value as or more extreme than the value we got — the area to the right of the realized value of \((TS)\).

Star-gazing is often silly; where’s the QOI?

\(\Rightarrow\) Can use likelihood: to compute tests and \(p\)-values.
What’s the best theory of inference?

1. Likelihood?
   - Bayes?
   - Neyman-Pearson?
   - Criteria estimators?
   - Finite or asymptotic based theory?
   - Decision theory?
   - Nonparametrics?
   - Semiparametrics?
   - Conditional inference?
   - Superpopulation-based inference?
   - etc.

2. None of these.

3. The right theory of inference: utilitarianism

Methods for applied researchers: either useful or irrelevant
What’s the best theory of inference?

1. Likelihood?
What’s the best theory of inference?

1. Likelihood? Bayes?

2. None of these.

3. The right theory of inference: utilitarianism?

Methods for applied researchers: either useful or irrelevant
What’s the best theory of inference?

1. Likelihood? Bayes? Neyman-Pearson?
What’s the best theory of inference?


2. No one of these.

3. The right theory of inference: utilitarianism.

Methods for applied researchers: either useful or irrelevant.
What’s the best theory of inference?

What’s the best theory of inference?

What’s the best theory of inference?

What’s the best theory of inference?

What’s the best theory of inference?

What’s the best theory of inference?

What’s the best theory of inference?

What’s the best theory of inference?


2. No
What’s the best theory of inference?


2. None
What’s the best theory of inference?


2. None of
What’s the best theory of inference?


2. None of these.
What’s the best theory of inference?


2. None of these.

3. The right theory of inference:
What’s the best theory of inference?


2. None of these.

3. The right theory of inference: utilitarianism
What’s the best theory of inference?


2. None of these.

3. The right theory of inference: utilitarianism

4. Methods for applied researchers: either useful or irrelevant
Unification of Theories of Inference

• Can't bank on agreement on normative issues!
• Even if there is agreement, it won't hold or shouldn't
• Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
  - Various types of robust or semi-parametric methods
  - Matching for use as preprocessing for parametric analysis
  - Bayesian model averaging, with a large enough class of models to average over
  - Committee methods, mixture of experts models
  - Models with highly flexible functional forms

• The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them
Unification of Theories of Inference

• Can’t bank on agreement on normative issues!
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
  - Various types of robust or semi-parametric methods
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
  - Various types of robust or semi-parametric methods
  - Matching for use as preprocessing for parametric analysis
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
  - Various types of robust or semi-parametric methods
  - Matching for use as preprocessing for parametric analysis
  - Bayesian model averaging, with a large enough class of models to average over
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
  - Various types of robust or semi-parametric methods
  - Matching for use as preprocessing for parametric analysis
  - Bayesian model averaging, with a large enough class of models to average over
  - Committee methods, mixture of experts models
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
  - Various types of robust or semi-parametric methods
  - Matching for use as preprocessing for parametric analysis
  - Bayesian model averaging, with a large enough class of models to average over
  - Committee methods, mixture of experts models
  - Models with highly flexible functional forms
Unification of Theories of Inference

- Can’t bank on agreement on normative issues!
- Even if there is agreement, it won’t hold or shouldn’t
- Alternative convergence is occurring: different methods giving the same result.
  - Likelihood or Bayes with careful goodness of fit checks
  - Various types of robust or semi-parametric methods
  - Matching for use as preprocessing for parametric analysis
  - Bayesian model averaging, with a large enough class of models to average over
  - Committee methods, mixture of experts models
  - Models with highly flexible functional forms
- The key: No assumptions can be trusted; all theories of inference condition on assumptions and so data analysts always struggle trying to understand and get around them
The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{stn}(y_i | \mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall i \neq j$.

Derive the full probability density of all $y$, $Pr(data | model)$

$$P(y|\mu) \equiv P(y_1, \ldots, y_n | \mu_1, \ldots, \mu_n) = n \prod_{i=1}^{n} f_{stn}(y_i | \mu_i) = n \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \mu_i)^2}{2}\right)$$

reparameterizing with $\mu_i = \beta$:

$$P(y|\beta) \equiv P(y_1, \ldots, y_n | \beta) = n \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \beta)^2}{2}\right)$$

Quiz: What can you do with this probability density?
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{stn}(y_i | \mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall i \neq j$.

Derive the full probability density of all $y$, $Pr(data|model) = P(y|\mu) = \prod_{i=1}^{n} f_{stn}(y_i | \mu_i) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp(- (y_i - \mu_i)^2)$

reparameterizing with $\mu_i = \beta$:

$P(y|\beta) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp(- (y_i - \beta)^2)$

Quiz: What can you do with this probability density?
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{stn}(y_i|\mu_i)$, normal stochastic component
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{stn}(y_i|\mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{\text{stn}}(y_i|\mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall i \neq j$. 
The Model

1. $Y_i \sim f_{stn}(y_i|\mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall i \neq j$.

Derive the full probability density of all $y$, $\Pr(\text{data}|\text{model})$
A Simple Likelihood Model: Stylized Normal, no X

The Model

1. $Y_i \sim f_{stn}(y_i|\mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall i \neq j$.

Derive the full probability density of all $y$, $Pr(\text{data|model})$

$$P(y|\mu) = P(y_1, \ldots, y_n|\mu_1, \ldots, \mu_n) = \prod_{i=1}^{n} f_{stn}(y_i|\mu_i)$$
A Simple Likelihood Model: Stylized Normal, no X

The Model

1. \( Y_i \sim f_{stn}(y_i|\mu_i) \), normal stochastic component
2. \( \mu_i = \beta \), a constant systematic component (no covariates)
3. \( Y_i \) and \( Y_j \) are independent \( \forall \ i \neq j \).

Derive the full probability density of all \( y \), \( \Pr(\text{data}|\text{model}) \)

\[
P(y|\mu) = \prod_{i=1}^{n} f_{stn}(y_i|\mu_i) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \mu_i)^2}{2}\right)
\]
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{stn}(y_i|\mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall i \neq j$.

Derive the full probability density of all $y$, $Pr(data|model)$

$$P(y|\mu) = P(y_1, \ldots, y_n|\mu_1, \ldots, \mu_n) = \prod_{i=1}^{n} f_{stn}(y_i|\mu_i)$$

$$= \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(\frac{-(y_i - \mu_i)^2}{2}\right)$$

reparameterizing with $\mu_i = \beta$:
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{stn}(y_i|\mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall \ i \neq j$.

Derive the full probability density of all $y$, $\Pr(\text{data}|\text{model})$

$$P(y|\mu) = P(y_1, \ldots, y_n|\mu_1, \ldots, \mu_n) = \prod_{i=1}^{n} f_{stn}(y_i|\mu_i)$$

$$= \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \mu_i)^2}{2}\right)$$

reparameterizing with $\mu_i = \beta$:

$$P(y|\beta) = P(y_1, \ldots, y_n|\beta) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \beta)^2}{2}\right)$$

Quiz: What can you do with this probability density?
A Simple Likelihood Model: Stylized Normal, no $X$

The Model

1. $Y_i \sim f_{\text{stn}}(y_i|\mu_i)$, normal stochastic component
2. $\mu_i = \beta$, a constant systematic component (no covariates)
3. $Y_i$ and $Y_j$ are independent $\forall i \neq j$.

Derive the full probability density of all $y$, $Pr(\text{data}|\text{model})$

$$P(y|\mu) \equiv P(y_1, \ldots, y_n|\mu_1, \ldots, \mu_n) = \prod_{i=1}^{n} f_{\text{stn}}(y_i|\mu_i)$$

$$= \prod_{i=1}^{n} (2\pi)^{-1/2} \exp \left( -\frac{(y_i - \mu_i)^2}{2} \right)$$

reparameterizing with $\mu_i = \beta$:

$$P(y|\beta) \equiv P(y_1, \ldots, y_n|\beta) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp \left( -\frac{(y_i - \beta)^2}{2} \right)$$

Quiz: What can you do with this probability density?
Derive the Log-Likelihood

The likelihood of \( \beta \) having generated the data we observe:

\[
L(\beta | y) = k(y) \prod_{i=1}^{n} f_{stn}(y_i | \beta)
\]

The log-likelihood (Recall: \( \ln(ab) = \ln(a) + \ln(b) \)):

\[
\ln L(\beta | y) = \ln[k(y)] + n \sum_{i=1}^{n} \ln f_{stn}(y_i | \beta) = \ln[k(y)] + n \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - n \sum_{i=1}^{n} \frac{1}{2}(y_i - \beta)^2 \\
\approx n \sum_{i=1}^{n} -\frac{1}{2}(y_i - \beta)^2
\]

Quiz: What subs for \( \beta \) to make \( \ln L \) the largest? What's that called?
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y)^n \prod_{i=1}^{n} f_{stn}(y_i|\beta)$$

The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$\ln L(\beta|y) = \ln[k(y)] + n \sum_{i=1}^{n} \ln f_{stn}(y_i|\beta) = \ln[k(y)] + n \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - n \sum_{i=1}^{1/2}(y_i - \beta)^2 \approx n \sum_{i=1}^{1/2}(y_i - \beta)^2$$

Quiz: What subs for $\beta$ to make $\ln L$ the largest? What's that called?
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{stn}(y_i|\beta)$$
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta)$$

$$= k(y) \prod_{i=1}^{n} (2\pi)^{-\frac{1}{2}} \exp\left(\frac{-(y_i - \beta)^2}{2}\right)$$

Quiz: What subs for $\beta$ to make $\ln L$ the largest? What's that called?
Derive the Log-Likelihood

The *likelihood* of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{stn}(y_i|\beta)$$

$$= k(y) \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \beta)^2}{2}\right)$$

The **log-likelihood** (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

Quiz: What subs for $\beta$ to make $\ln L$ the largest? What's that called?
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta)$$

$$= k(y) \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(\frac{-(y_i - \beta)^2}{2}\right)$$

The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$\ln L(\beta|y) = \ln[k(y)] + \sum_{i=1}^{n} \ln f_{\text{stn}}(y_i|\beta)$$
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{stn}(y_i|\beta)$$

$$= k(y) \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(\frac{-(y_i - \beta)^2}{2}\right)$$

The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$\ln L(\beta|y) = \ln[k(y)] + \sum_{i=1}^{n} \ln f_{stn}(y_i|\beta)$$

$$= \ln[k(y)] + \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n} \frac{1}{2}(y_i - \beta)^2$$

Quiz: What subs for $\beta$ to make $\ln L$ the largest? What's that called?
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta)$$

$$= k(y) \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \beta)^2}{2}\right)$$

The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$\ln L(\beta|y) = \ln[k(y)] + \sum_{i=1}^{n} \ln f_{\text{stn}}(y_i|\beta)$$

$$= \ln[k(y)] + \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n} \frac{1}{2}(y_i - \beta)^2$$

$$= \sum_{i=1}^{n} -\frac{1}{2}(y_i - \beta)^2$$
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta)$$

$$= k(y) \prod_{i=1}^{n} (2\pi)^{-1/2} \exp \left( -\frac{(y_i - \beta)^2}{2} \right)$$

The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$\ln L(\beta|y) = \ln[k(y)] + \sum_{i=1}^{n} \ln f_{\text{stn}}(y_i|\beta)$$

$$= \ln[k(y)] + \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n} \frac{1}{2}(y_i - \beta)^2$$

$$= \sum_{i=1}^{n} -\frac{1}{2}(y_i - \beta)^2 = -\frac{1}{2} \sum_{i=1}^{n} (y_i - \beta)^2$$
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$ L(\beta|y) = k(y) \prod_{i=1}^{n} f_{\text{stn}}(y_i|\beta) $$

$$ = k(y) \prod_{i=1}^{n} (2\pi)^{-1/2} \exp \left( \frac{-(y_i - \beta)^2}{2} \right) $$

The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$ \ln L(\beta|y) = \ln[k(y)] + \sum_{i=1}^{n} \ln f_{\text{stn}}(y_i|\beta) $$

$$ = \ln[k(y)] + \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n} \frac{1}{2}(y_i - \beta)^2 $$

$$ = \sum_{i=1}^{n} -\frac{1}{2}(y_i - \beta)^2 = -\frac{1}{2} \sum_{i=1}^{n} (y_i - \beta)^2 $$

Quiz: What subs for $\beta$ to make $\ln L$ the largest?
Derive the Log-Likelihood

The likelihood of $\beta$ having generated the data we observe:

$$L(\beta|y) = k(y) \prod_{i=1}^{n} f_{stn}(y_i|\beta)$$

$$= k(y) \prod_{i=1}^{n} (2\pi)^{-1/2} \exp\left(-\frac{(y_i - \beta)^2}{2}\right)$$

The log-likelihood (Recall: $\ln(ab) = \ln(a) + \ln(b)$):

$$\ln L(\beta|y) = \ln[k(y)] + \sum_{i=1}^{n} \ln f_{stn}(y_i|\beta)$$

$$= \ln[k(y)] + \sum_{i=1}^{n} \ln[(2\pi)^{-1/2}] - \sum_{i=1}^{n} \frac{1}{2}(y_i - \beta)^2$$

$$= \sum_{i=1}^{n} -\frac{1}{2}(y_i - \beta)^2 = -\frac{1}{2} \sum_{i=1}^{n} (y_i - \beta)^2$$

Quiz: What subs for $\beta$ to make $\ln L$ the largest? What’s that called?
Log-likelihood interpretation

1. The log-likelihood is quadratic (multiply out the expression).
2. This curve summarizes all information the data gives about $\beta$, assuming the model.
3. The maximum is at the same point as the least squares point.
4. The MLE is at the same point as the MVLUE.
5. No reason to summarize this curve with only the MLE.
1. The log-likelihood is quadratic (multiply out the expression)
Log-likelihood interpretation

1. The log-likelihood is quadratic (multiply out the expression)
2. This curve summarizes all information the data gives about $\beta$, assuming the model.
1. The log-likelihood is quadratic (multiply out the expression)
2. This curve summarizes all information the data gives about $\beta$, assuming the model.
3. The maximum is at the same point as the least squares point.
1. The log-likelihood is quadratic (multiply out the expression)
2. This curve summarizes all information the data gives about $\beta$, assuming the model.
3. The maximum is at the same point as the least squares point
4. The MLE is at the same point as the MVLUE
Log-likelihood interpretation

1. The log-likelihood is quadratic (multiply out the expression)
2. This curve summarizes all information the data gives about $\beta$, assuming the model.
3. The maximum is at the same point as the least squares point
4. The MLE is at the same point as the MVLUE
5. No reason to summarize this curve with only the MLE
Summarizing \( k \)-dimensional space
Summarizing $k$-dimensional space

- Graphs
Summarizing $k$-dimensional space

- Graphs
- The problem of Flatland
Summarizing $k$-dimensional space

- Graphs
- The problem of Flatland
- The curse of dimensionality
Summarizing $k$-dimensional space

- Graphs
- The problem of Flatland
- The curse of dimensionality
- We’ll often use:
Summarizing $k$-dimensional space

- Graphs
- The problem of Flatland
- The curse of dimensionality
- We’ll often use:
  - $\hat{\beta}$, a vector of point estimates, the MLE
Summarizing $k$-dimensional space

- Graphs
- The problem of Flatland
- The curse of dimensionality
- We’ll often use:
  - $\hat{\beta}$, a vector of point estimates, the MLE
  - Curvature at the maximum (standard errors, about which more shortly)
How to find the maximum?

1. Analytically — sometimes possible
   - Take derivative of \( \ln L(\theta | y) \) w.r.t. \( \theta \)
   - Set to 0, substituting \( \hat{\theta} \) for \( \theta \)
   - If possible, solve for \( \theta \), and label it \( \hat{\theta} \)
   - Check second order condition: make sure second derivative w.r.t. \( \theta \) is negative (so it's a maximum rather than a minimum)

2. Numerically — let the computer do the work for you
   - We'll show you how

(Sound good?)
How to find the maximum?

Goal: Find the value of $\theta = \{\theta_1, \ldots, \theta_k\}$ that maximizes $L(\theta|y)$.
How to find the maximum?

Goal: Find the value of $\theta = \{\theta_1, \ldots, \theta_k\}$ that maximizes $L(\theta|y)$

1. **Analytically** — sometimes possible
How to find the maximum?

Goal: Find the value of $\theta = \{\theta_1, \ldots, \theta_k\}$ that maximizes $L(\theta|y)$

1. **Analytically** — sometimes possible
   - Take derivative of $\ln L(\theta|y)$ w.r.t. $\theta$

   $$\frac{\partial}{\partial \theta} \ln L(\theta|y) = 0$$
   - If possible, solve for $\theta$, and label it $\hat{\theta}$
   - Check second order condition: make sure second derivative w.r.t. $\theta$ is negative (so it's a maximum rather than a minimum)
How to find the maximum?
Goal: Find the value of \( \theta = \{\theta_1, \ldots, \theta_k\} \) that maximizes \( L(\theta|y) \)

1. **Analytically** — sometimes possible
   - Take derivative of \( \ln L(\theta|y) \) w.r.t. \( \theta \)
   - Set to 0, substituting \( \hat{\theta} \) for \( \theta \)
How to find the maximum?

Goal: Find the value of \( \theta = \{\theta_1, \ldots, \theta_k\} \) that maximizes \( L(\theta|y) \)

1. **Analytically** — sometimes possible
   - Take derivative of \( \ln L(\theta|y) \) w.r.t. \( \theta \)
   - Set to 0, substituting \( \hat{\theta} \) for \( \theta \)

\[
\left. \frac{\partial \ln L(\theta|y)}{\partial \theta} \right|_{\theta=\hat{\theta}} = 0
\]
How to find the maximum?

Goal: Find the value of $\theta = \{\theta_1, \ldots, \theta_k\}$ that maximizes $L(\theta|y)$

1. **Analytically** — sometimes possible
   - Take derivative of $\ln L(\theta|y)$ w.r.t. $\theta$
   - Set to 0, substituting $\hat{\theta}$ for $\theta$

   $$\left| \frac{\partial \ln L(\theta|y)}{\partial \theta} \right|_{\theta=\hat{\theta}} = 0$$

   - If possible, solve for $\theta$, and label it $\hat{\theta}$
How to find the maximum?

Goal: Find the value of \( \theta = \{\theta_1, \ldots, \theta_k\} \) that maximizes \( L(\theta|y) \)

1. **Analytically** — sometimes possible
   - Take derivative of \( \ln L(\theta|y) \) w.r.t. \( \theta \)
   - Set to 0, substituting \( \hat{\theta} \) for \( \theta \)

\[
\left| \frac{\partial \ln L(\theta|y)}{\partial \theta} \right|_{\theta=\hat{\theta}} = 0
\]

   - If possible, solve for \( \theta \), and label it \( \hat{\theta} \)
   - Check second order condition: make sure second derivative w.r.t. \( \theta \) is negative (so its a maximum rather than a minimum)
How to find the maximum?
Goal: Find the value of $\theta = \{\theta_1, \ldots, \theta_k\}$ that maximizes $L(\theta|y)$

1. **Analytically** — sometimes possible
   - Take derivative of $\ln L(\theta|y)$ w.r.t. $\theta$
   - Set to 0, substituting $\hat{\theta}$ for $\theta$
   
   $$\left| \frac{\partial \ln L(\theta|y)}{\partial \theta} \right|_{\theta=\hat{\theta}} = 0$$

   - If possible, solve for $\theta$, and label it $\hat{\theta}$
   - Check second order condition: make sure second derivative w.r.t. $\theta$ is negative (so its a maximum rather than a minimum)

2. **Numerically** — let the computer do the work for you
How to find the maximum?

Goal: Find the value of \( \theta = \{\theta_1, \ldots, \theta_k\} \) that maximizes \( L(\theta|y) \)

1. **Analytically** — sometimes possible
   - Take derivative of \( \ln L(\theta|y) \) w.r.t. \( \theta \)
   - Set to 0, substituting \( \hat{\theta} \) for \( \theta \)

   \[
   \left| \frac{\partial \ln L(\theta|y)}{\partial \theta} \right|_{\theta=\hat{\theta}} = 0
   \]

   - If possible, solve for \( \theta \), and label it \( \hat{\theta} \)
   - Check second order condition: make sure second derivative w.r.t. \( \theta \) is negative (so its a maximum rather than a minimum)

2. **Numerically** — let the computer do the work for you
   - We’ll show you how
How to find the maximum?

Goal: Find the value of $\theta = \{\theta_1, \ldots, \theta_k\}$ that maximizes $L(\theta | y)$

1. **Analytically** — sometimes possible
   - Take derivative of $\ln L(\theta | y)$ w.r.t. $\theta$
   - Set to 0, substituting $\hat{\theta}$ for $\theta$
   
   $$\left| \frac{\partial \ln L(\theta | y)}{\partial \theta} \right|_{\theta = \hat{\theta}} = 0$$

   - If possible, solve for $\theta$, and label it $\hat{\theta}$
   - Check second order condition: make sure second derivative w.r.t. $\theta$ is negative (so its a maximum rather than a minimum)

2. **Numerically** — let the computer do the work for you
   - We’ll show you how
   - (Sound good?)
 Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - **Unbiasedness:**
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} \cdot n \cdot \mu = \mu$
   - **Minimum variance ("efficiency"):**
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V \left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{\sigma^2}{n}$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs: estimate $\sigma^2$ with $\hat{\sigma}^2$ or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   - Not true for other methods of inference: e.g. $E(\bar{y}) = \mu$.
   - What is an unbiased estimate of $1/\mu$? Is it $1/\bar{y}$? No: $E(1/\bar{y}) \neq 1/E(\bar{y})$
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)

- **Unbiasedness**
  - **Definition**
    \[
    
    E(\hat{\theta}) = \theta
    \]
  - **Example**
    \[
    E(\overline{Y}) = \overline{E(\sum_{i=1}^{n} Y_i)} = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu
    \]

- **Minimum variance ("efficiency")**
  - **Variance to be minimized**
    \[
    V(\hat{\theta})
    \]
  - **Example**
    \[
    V(\overline{Y}) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \frac{\sigma^2}{n}
    \]
  - **Efficiency**
    Define \( \hat{\theta} \) to minimize \( V(\hat{\theta}) \), s.t. \( E(\hat{\theta}) = \theta \)

- If there is a MVUE, ML will find it
- If there isn't one, ML will still usually find a good estimator

2. Invariance to Reparameterization

- Both are MLEs:
  - Estimate \( \sigma^2 \) with \( \hat{\sigma}^2 \) or estimate \( \sigma \) with \( \hat{\sigma} \) and calculate \( \hat{\sigma}^2 \)

- Not true for other methods of inference:
  - e.g. \( E(\overline{y}) = \mu \).

  - What is an unbiased estimate of \( 1/\mu \)?
    - Is it \( 1/\overline{y} \)?
    - No:
      \[
      E(1/\overline{y}) \neq 1/E(\overline{y})
      \]
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     \[ \mathbb{E}(\hat{\theta}) = \theta \]
     \[ \mathbb{E}(\bar{Y}) = \frac{1}{n} \sum_{i=1}^{n} Y_i = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(Y_i) = \frac{1}{n} n \mu = \mu \]
   - **Minimum variance (“efficiency”)**
     \[ \text{Variance to be minimized: } \mathbb{V}(\hat{\theta}) \]
     \[ \mathbb{V}(\bar{Y}) = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{V}(Y_i) = \frac{1}{n^2} n \sigma^2 = \frac{\sigma^2}{n} \]
     \[ \text{Efficiency: Define } \hat{\theta} \text{ to minimize } \mathbb{V}(\hat{\theta}), \text{ s.t. } \mathbb{E}(\hat{\theta}) = \theta \]
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. **Invariance to Reparameterization**
   - Both are MLEs:
     \[ \hat{\sigma}^2 \] or \[ \hat{\sigma} \] and calculate \[ \hat{\sigma}^2 \]
   - Not true for other methods of inference:
     \[ \mathbb{E}(\bar{y}) = \mu \].
     What is an unbiased estimate of \(1/\mu\)?
     Is it \(\frac{1}{\bar{y}}\)?
     Nope: \[ \mathbb{E}(1/\bar{y}) \neq 1/\mathbb{E}(\bar{y}) \]

3. **Invariance to sampling plans**
   - OK to look at results while deciding how much data to collect
   - In fact, it’s a great idea! (e.g., King, Schneer, White 2017)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = $
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   • Unbiasedness:
     • Definition: $E(\hat{\theta}) = \theta$
     • Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) =$
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) =$

2. **Invariance to Reparameterization**
   - Both are MLEs:
     - Estimate $\sigma^2$ with $\hat{\sigma}^2$ or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   - Not true for other methods of inference: e.g. $E(\bar{y}) = \mu$.
   - What is an unbiased estimate of $1/\mu$? Is it $1/\bar{y}$? No:
     - $E(1/\bar{y}) \neq 1/E(\bar{y})$

3. **Invariance to sampling plans**
   - OK to look at results while deciding how much data to collect
   - In fact, it's a great idea! (e.g., King, Schneer, White 2017)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu$
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu$
   - Minimum variance ("efficiency")
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu$
   - Minimum variance (“efficiency”)
     - Variance to be minimized: $V(\hat{\theta})$
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - **Definition:** \( E(\hat{\theta}) = \theta \)
     - **Example:** \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu \)
   - **Minimum variance (“efficiency”)**
     - **Variance to be minimized:** \( V(\hat{\theta}) \)
     - **Example:** \( V(\bar{Y}) = \)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: \( E(\hat{\theta}) = \theta \)
     - Example: \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu \)
   - Minimum variance (“efficiency”)
     - Variance to be minimized: \( V(\hat{\theta}) \)
     - Example: \( V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \)

   \[ \frac{\sigma^2}{n} \]
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: \( E(\hat{\theta}) = \theta \)
     - Example: 
       
       \[
       E(\bar{Y}) = E\left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu 
       \]

   - Minimum variance ("efficiency")
     - Variance to be minimized: \( V(\hat{\theta}) \)
     - Example: 
       
       \[
       V(\bar{Y}) = V\left( \frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = 
       \]
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
   - Minimum variance ("efficiency")
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \ldots$
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu$
   - **Minimum variance ("efficiency")**
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n$
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
   - **Minimum variance ("efficiency")**
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \sigma^2 / n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu$
   - Minimum variance (“efficiency”)
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   - If there is a MVUE, ML will find it
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: \( E(\hat{\theta}) = \theta \)
     - Example: \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu \)
   - **Minimum variance ("efficiency")**
     - Variance to be minimized: \( V(\hat{\theta}) \)
     - Example: \( V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n \)
     - Efficiency: Define \( \hat{\theta} \) to minimize \( V(\hat{\theta}) \), s.t. \( E(\hat{\theta}) = \theta \)
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. **Invariance to Reparameterization**
   - Both are MLEs:
     - Estimate \( \sigma^2 \) with \( \hat{\sigma}^2 \)
     - Estimate \( \sigma \) with \( \hat{\sigma} \) and calculate \( \hat{\sigma}^2 \)
   - Not true for other methods of inference:
     - e.g. \( E(\bar{y}) = \mu \).
     - What is an unbiased estimate of \( 1/\mu \)?
     - Is it \( 1/\bar{y} \)?
     - Nope: \( E(1/\bar{y}) \neq 1/E(\bar{y}) \)

3. **Invariance to sampling plans**
   - OK to look at results while deciding how much data to collect
   - In fact, it’s a great idea! (e.g., King, Schneer, White 2017)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
   - Minimum variance (“efficiency”)
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \sigma^2/n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
     - If there is a MVUE, ML will find it
     - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization

   - Both are MLEs:
     - Estimate $\sigma^2$ with $\hat{\sigma}^2$ or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   - Not true for other methods of inference:
     - e.g. $E(\bar{y}) = \mu$.
     - What is an unbiased estimate of $1/\mu$? Is it $1/\bar{y}$? No:
       $E\left(\frac{1}{\bar{y}}\right) \neq \frac{1}{E(\bar{y})}$

2. Invariance to sampling plans
   - OK to look at results while deciding how much data to collect
   - In fact, it’s a great idea! (e.g., King, Schneer, White 2017)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu$
   - Minimum variance (“efficiency”)
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
       - If there is a MVUE, ML will find it
       - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs:
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: \( E(\hat{\theta}) = \theta \)
     - Example: \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu \)
   - **Minimum variance (“efficiency”)**
     - Variance to be minimized: \( V(\hat{\theta}) \)
     - Example: \( V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n \)
     - Efficiency: Define \( \hat{\theta} \) to minimize \( V(\hat{\theta}) \), s.t. \( E(\hat{\theta}) = \theta \)
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. **Invariance to Reparameterization**
   - Both are MLEs: Estimate \( \sigma^2 \) with \( \hat{\sigma}^2 \)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
   - Minimum variance (“efficiency”)
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \sigma^2/n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs: Estimate $\sigma^2$ with $\hat{\sigma}^2$
     or
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: \( E(\hat{\theta}) = \theta \)
     - Example: \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu \)
   - **Minimum variance (“efficiency”)**
     - Variance to be minimized: \( V(\hat{\theta}) \)
     - Example: \( V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n \)
     - Efficiency: Define \( \hat{\theta} \) to minimize \( V(\hat{\theta}) \), s.t. \( E(\hat{\theta}) = \theta \)
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. **Invariance to Reparameterization**
   - Both are MLEs: Estimate \( \sigma^2 \) with \( \hat{\sigma}^2 \)
     or estimate \( \sigma \) with \( \hat{\sigma} \) and calculate \( \hat{\sigma}^2 \)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: \( E(\hat{\theta}) = \theta \)
     - Example: \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu \)
   - Minimum variance ("efficiency")
     - Variance to be minimized: \( V(\hat{\theta}) \)
     - Example: \( V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n \)
     - Efficiency: Define \( \hat{\theta} \) to minimize \( V(\hat{\theta}) \), s.t. \( E(\hat{\theta}) = \theta \)
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs: Estimate \( \sigma^2 \) with \( \hat{\sigma}^2 \)
     or estimate \( \sigma \) with \( \hat{\sigma} \) and calculate \( \hat{\sigma}^2 \)
   - Not true for other methods of inference:
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
   - Minimum variance (“efficiency”)
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \sigma^2 / n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs: Estimate $\sigma^2$ with $\hat{\sigma}^2$
     or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   - Not true for other methods of inference: e.g. $E(\bar{y}) = \mu$. 

What is an unbiased estimate of $1/\mu$?
Is it $1/\bar{y}$?
Nope: $E(1/\bar{y}) \neq 1/E(\bar{y})$
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   • Unbiasedness:
     • Definition: \( E(\hat{\theta}) = \theta \)
     • Example: \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu \)
   • Minimum variance ("efficiency")
     • Variance to be minimized: \( V(\hat{\theta}) \)
     • Example: \( V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n \)
     • Efficiency: Define \( \hat{\theta} \) to minimize \( V(\hat{\theta}) \), s.t. \( E(\hat{\theta}) = \theta \)
   • If there is a MVUE, ML will find it
   • If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   • Both are MLEs: Estimate \( \sigma^2 \) with \( \hat{\sigma}^2 \)
     or estimate \( \sigma \) with \( \hat{\sigma} \) and calculate \( \hat{\sigma}^2 \)
   • Not true for other methods of inference: e.g. \( E(\bar{y}) = \mu \). What is an unbiased estimate of \( 1/\mu \)?
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu$
   - Minimum variance ("efficiency")
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs: Estimate $\sigma^2$ with $\hat{\sigma}^2$
     or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   - Not true for other methods of inference: e.g. $E(\bar{y}) = \mu$. What is an unbiased estimate of $1/\mu$? Is it $1/\bar{y}$?
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: \( E(\hat{\theta}) = \theta \)
     - Example: \( E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu \)
   - Minimum variance (“efficiency”)
     - Variance to be minimized: \( V(\hat{\theta}) \)
     - Example: \( V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n \)
     - Efficiency: Define \( \hat{\theta} \) to minimize \( V(\hat{\theta}) \), s.t. \( E(\hat{\theta}) = \theta \)
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs: Estimate \( \sigma^2 \) with \( \hat{\sigma}^2 \)
     or estimate \( \sigma \) with \( \hat{\sigma} \) and calculate \( \hat{\sigma}^2 \)
   - Not true for other methods of inference: e.g. \( E(\bar{y}) = \mu \). What is an unbiased estimate of \( 1/\mu \)? Is it \( 1/\bar{y} \)? Nope: \( E(1/\bar{y}) \neq 1/E(\bar{y}) \)
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   - Unbiasedness:
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
   - Minimum variance (“efficiency”)
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \sigma^2 / n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   - Both are MLEs: Estimate $\sigma^2$ with $\hat{\sigma}^2$
     or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   - Not true for other methods of inference: e.g. $E(\bar{y}) = \mu$. What is an unbiased estimate of $1/\mu$? Is it $1/\bar{y}$? Nope: $E(1/\bar{y}) \neq 1/E(\bar{y})$

3. Invariance to sampling plans
Finite Sample Properties of the MLE

1. Minimum variance unbiased estimator (MVUE)
   
   • Unbiasedness:
     
     • Definition: $E(\hat{\theta}) = \theta$
     
     • Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n \mu = \mu$
   
   • Minimum variance ("efficiency")
     
     • Variance to be minimized: $V(\hat{\theta})$
     
     • Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n \sigma^2 = \sigma^2/n$
     
     • Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   
   • If there is a MVUE, ML will find it
   
   • If there isn’t one, ML will still usually find a good estimator

2. Invariance to Reparameterization
   
   • Both are MLEs: Estimate $\sigma^2$ with $\hat{\sigma}^2$
     
     or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   
   • Not true for other methods of inference: e.g. $E(\bar{y}) = \mu$. What is an unbiased estimate of $1/\mu$? Is it $1/\bar{y}$? Nope: $E(1/\bar{y}) \neq 1/E(\bar{y})$

3. Invariance to sampling plans
   
   • OK to look at results while deciding how much data to collect
Finite Sample Properties of the MLE

1. **Minimum variance unbiased estimator (MVUE)**
   - **Unbiasedness:**
     - Definition: $E(\hat{\theta}) = \theta$
     - Example: $E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} n\mu = \mu$
   - **Minimum variance (“efficiency”)**
     - Variance to be minimized: $V(\hat{\theta})$
     - Example: $V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^{n} Y_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} V(Y_i) = \frac{1}{n^2} n\sigma^2 = \sigma^2/n$
     - Efficiency: Define $\hat{\theta}$ to minimize $V(\hat{\theta})$, s.t. $E(\hat{\theta}) = \theta$
   - If there is a MVUE, ML will find it
   - If there isn’t one, ML will still usually find a good estimator

2. **Invariance to Reparameterization**
   - Both are MLEs: Estimate $\sigma^2$ with $\hat{\sigma}^2$
     or estimate $\sigma$ with $\hat{\sigma}$ and calculate $\hat{\sigma}^2$
   - Not true for other methods of inference: e.g. $E(\bar{y}) = \mu$. What is an unbiased estimate of $1/\mu$? Is it $1/\bar{y}$? Nope: $E(1/\bar{y}) \neq 1/E(\bar{y})$

3. **Invariance to sampling plans**
   - OK to look at results while deciding how much data to collect
   - In fact, it’s a great idea! (e.g., King, Schneer, White 2017)
Asymptotic Properties of the MLE

1. Consistency (from the Law of Large Numbers)
   - As $n \rightarrow \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value.
   - Why do we care? An approximation to: more data helps.

2. Asymptotic normality (from the Central Limit Theorem)
   - As $n \rightarrow \infty$, repeated samples of MLE/se(MLE) converge to Normal.
   - Why do we care? If $N$ is large enough, the asymptotic distribution is a good approximation.

3. Asymptotic efficiency
   - As $n \rightarrow \infty$, MLE contains as much information as can be packed into a point estimator; it is the MVUE.
   - Why do we care? If $n$ is large enough, we’re not wasting data.

Quiz: Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?
Asymptotic Properties of the MLE


   • As $n \rightarrow \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value.

   Why do we care?

   An approximation to: more data helps.
Asymptotic Properties of the MLE

1. **Consistency** *(from the Law of Large Numbers).*
   - As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
Asymptotic Properties of the MLE

1. **Consistency** (from the Law of Large Numbers).
   - As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
   - **Why do we care?**
Asymptotic Properties of the MLE

1. **Consistency** (from the Law of Large Numbers).
   - As \( n \to \infty \), the sampling distribution of the MLE collapses to a spike over the parameter value
   - **Why do we care?** An approximation to: more data helps

2. **Asymptotic normality** (from the Central Limit Theorem):
   - As \( n \to \infty \), repeated samples of MLE/se(MLE) converge to Normal
   - **Why do we care?** If \( N \) is large enough, the asymptotic distribution is a good approximation

3. **Asymptotic efficiency**
   - As \( n \to \infty \), MLE contains as much information as can be packed into a point estimator; it is the MVUE
   - **Why do we care?** If \( n \) is large enough, we’re not wasting data
Asymptotic Properties of the MLE

1. **Consistency** (from the Law of Large Numbers).
   - As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
   - **Why do we care?** An approximation to: more data helps

2. **Asymptotic normality** (from the Central Limit Theorem):
Asymptotic Properties of the MLE

   - As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
   - Why do we care? An approximation to: more data helps

2. Asymptotic normality (from the Central Limit Theorem):
   - As $n \to \infty$, repeated samples of MLE/se(MLE) converge to Normal
Asymptotic Properties of the MLE

1. **Consistency** (from the Law of Large Numbers).
   - As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
   - **Why do we care?** An approximation to: more data helps

2. **Asymptotic normality** (from the Central Limit Theorem):
   - As $n \to \infty$, repeated samples of MLE/se(MLE) converge to Normal
   - **Why do we care?**
Asymptotic Properties of the MLE

   • As \( n \to \infty \), the sampling distribution of the MLE collapses to a spike over the parameter value
   • Why do we care? An approximation to: more data helps

2. Asymptotic normality (from the Central Limit Theorem):
   • As \( n \to \infty \), repeated samples of MLE/se(MLE) converge to Normal
   • Why do we care? If \( N \) is large enough, the asymptotic distribution is a good approximation
Asymptotic Properties of the MLE

1. **Consistency** (from the Law of Large Numbers).
   - As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
   - **Why do we care?** An approximation to: more data helps

2. **Asymptotic normality** (from the Central Limit Theorem):
   - As $n \to \infty$, repeated samples of MLE/se(MLE) converge to Normal
   - **Why do we care?** If $N$ is large enough, the asymptotic distribution is a good approximation
   - **Quiz:** Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?
Asymptotic Properties of the MLE

1. **Consistency** (from the Law of Large Numbers).
   - As \( n \to \infty \), the sampling distribution of the MLE collapses to a spike over the parameter value
   - **Why do we care?** An approximation to: more data helps

2. **Asymptotic normality** (from the Central Limit Theorem):
   - As \( n \to \infty \), repeated samples of MLE/se(MLE) converge to Normal
   - **Why do we care?** If \( N \) is large enough, the asymptotic distribution is a good approximation
   - **Quiz:** Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?

3. **Asymptotic efficiency**
Asymptotic Properties of the MLE

   • As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value
   • Why do we care? An approximation to: more data helps

2. Asymptotic normality (from the Central Limit Theorem):
   • As $n \to \infty$, repeated samples of MLE/se(MLE) converge to Normal
   • Why do we care? If $N$ is large enough, the asymptotic distribution is a good approximation
   • Quiz: Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?

3. Asymptotic efficiency
   • As $n \to \infty$, MLE contains as much information as can be packed into a point estimator; it is the MVUE
Asymptotic Properties of the MLE

1. **Consistency** (from the Law of Large Numbers).
   - As \( n \to \infty \), the sampling distribution of the MLE collapses to a spike over the parameter value.
   - **Why do we care?** An approximation to: more data helps.

2. **Asymptotic normality** (from the Central Limit Theorem):
   - As \( n \to \infty \), repeated samples of MLE/se(MLE) converge to Normal.
   - **Why do we care?** If \( N \) is large enough, the asymptotic distribution is a good approximation.
   - **Quiz:** Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?

3. **Asymptotic efficiency**
   - As \( n \to \infty \), MLE contains as much information as can be packed into a point estimator; it is the MVUE.
   - **Why do we care?**
Asymptotic Properties of the MLE

1. **Consistency** *(from the Law of Large Numbers)*.
   - As $n \to \infty$, the sampling distribution of the MLE collapses to a spike over the parameter value.
   - *Why do we care?* An approximation to: more data helps.

2. **Asymptotic normality** *(from the Central Limit Theorem)*:
   - As $n \to \infty$, repeated samples of MLE/se(MLE) converge to Normal.
   - *Why do we care?* If $N$ is large enough, the asymptotic distribution is a good approximation.
   - *Quiz: Do the LLN and CLT (the 2 most important theorems in statistics) contradict each other?*

3. **Asymptotic efficiency**
   - As $n \to \infty$, MLE contains as much information as can be packed into a point estimator; it is the MVUE.
   - *Why do we care?* If $n$ is large enough, we’re not wasting data.
Sampling distributions of the MLE: CLT vs LLN
Sampling distributions of the MLE: CLT vs LLN

Why asymptotic approximations may work in small samples
Sampling distributions of the MLE: CLT vs LLN

Why asymptotic approximations may work in small samples
Quiz: Which is Unbiased & Inconsistent

\[
a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15
\]
biased,
inconsistent

\[
a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i
\]
unbiased,
inconsistent

\[
a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \frac{7}{n} \sum_{i=1}^{n} Y_i
\]
biased,
consistent

\[
a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i
\]
unbiased,
consistent (inefficient)

\[
a_5 = \frac{1}{n} \sum_{i=1}^{n} Y_i
\]
unbiased,
consistent, efficient
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \]
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \]

\[ a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i \]
Quiz: Which is Unbiased & Inconsistent

\[
a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15
\]

\[
a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i
\]

\[
a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} \frac{Y_i}{n}
\]
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \]

\[ a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i \]

\[ a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} Y_i/n \]

\[ a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i \]
Quiz: Which is Unbiased & Inconsistent

\[
a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15
\]
biased, inconsistent

\[
a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i
\]

\[
a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} Y_i/n
\]

\[
a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i
\]
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \]  
biased, inconsistent

\[ a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i \]  
unbiased, inconsistent

\[ a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} \frac{Y_i}{n} \]

\[ a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i \]
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \]
biased, inconsistent

\[ a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i \]
unbiased, inconsistent

\[ a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} Y_i/n \]
biased, consistent

\[ a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i \]
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \quad \text{biased, inconsistent} \]

\[ a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i \quad \text{unbiased, inconsistent} \]

\[ a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} Y_i/n \quad \text{biased, consistent} \]

\[ a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i \quad \text{unbiased, consistent (inefficient)} \]
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \quad \text{biased, inconsistent} \]

\[ a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i \quad \text{unbiased, inconsistent} \]

\[ a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} Y_i/n \quad \text{biased, consistent} \]

\[ a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i \quad \text{unbiased, consistent (inefficient)} \]

\[ a_5 = \frac{1}{n} \sum_{i=1}^{n} Y_i \]
Quiz: Which is Unbiased & Inconsistent

\[ a_1 = \frac{1}{n} \sum_{i=1}^{n} Y_i + 15 \]  
biased, inconsistent

\[ a_2 = \frac{1}{27} \sum_{i=1}^{27} Y_i \]  
unbiased, inconsistent

\[ a_3 = \frac{1}{n} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{7} Y_i/n \]  
biased, consistent

\[ a_4 = \frac{1}{n-2} \sum_{i=1}^{n-2} Y_i \]  
unbiased, consistent (inefficient)

\[ a_5 = \frac{1}{n} \sum_{i=1}^{n} Y_i \]  
unbiased, consistent, efficient
The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function
Three Measures of Uncertainty

- Relative heights at different parameter values: Likelihood Ratio
- Curvature at maximum: Standard Errors
- Slope at single parameter value: Rao's Score (LM)
Three Measures of Uncertainty

- Relative heights at different parameter values: Likelihood Ratio
- Curvature at maximum: Standard Errors
- Slope at single parameter value: Rao's Score (LM)

Uncertainty in Likelihood Inference
Three Measures of Uncertainty

- **Relative heights at different parameter values: Likelihood Ratio**
Three Measures of Uncertainty

- **Relative heights at different parameter values:** Likelihood Ratio
- **Curvature at maximum:** Standard Errors
Three Measures of Uncertainty

- Relative heights at different parameter values: Likelihood Ratio
- Curvature at maximum: Standard Errors
- Slope at single parameter value: Rao’s Score (LM)
Uncertainty via the Likelihood Ratio

Compare two likelihood models

- unrestricted model: \( L^* \)
- restricted (nested) model: \( L^* \)

Likelihood Ratio:
\[
L^* \geq L^* \implies L^* \leq 1
\]

Likelihood ratio: the ratio of 2 traditional probabilities

\[
L^* \equiv L(\theta_1|y) \propto k(y) P(y|\theta_1)
\]

\[
L(\theta_1|y) = \frac{k(y)}{k(y)} P(y|\theta_1)
\]

\[
L(\theta_2|y) = P(y|\theta_1)
\]

a risk ratio
Uncertainty via the Likelihood Ratio

- Compare two likelihood models
Uncertainty via the Likelihood Ratio

- Compare **two likelihood models**
  - unrestricted model: $L^*$
Uncertainty via the Likelihood Ratio

- Compare two likelihood models
  - unrestricted model: $L^*$
  - restricted (nested) model: $L^*_R$
Uncertainty via the Likelihood Ratio

• Compare two likelihood models
  • unrestricted model: $L^*$
  • restricted (nested) model: $L^*_R$
  • Likelihood Ratio:

\[
L^* \geq L^*_R \implies \frac{L^*_R}{L^*} \leq 1
\]
Uncertainty via the Likelihood Ratio

- Compare two likelihood models
  - unrestricted model: $L^*$
  - restricted (nested) model: $L_R^*$

- Likelihood Ratio:

\[ L^* \geq L_R^* \implies \frac{L_R^*}{L^*} \leq 1 \]

- Likelihood ratio: the ratio of 2 traditional probabilities
Uncertainty via the Likelihood Ratio

• Compare two likelihood models
  • unrestricted model: $L^*$
  • restricted (nested) model: $L_R^*$

• Likelihood Ratio:

\[
L^* \geq L_R^* \implies \frac{L_R^*}{L^*} \leq 1
\]

• Likelihood ratio: the ratio of 2 traditional probabilities

\[
L_R^* = L(\theta_1|y) \propto k(y)P(y|\theta_1)
\]
Uncertainty via the Likelihood Ratio

- Compare two likelihood models
  - unrestricted model: $L^*$
  - restricted (nested) model: $L_R^*$
- Likelihood Ratio:

$$L^* \geq L_R^* \implies \frac{L_R^*}{L^*} \leq 1$$

- Likelihood ratio: the ratio of 2 traditional probabilities

$$L_R^* = L(\theta_1 | y) \propto k(y)P(y | \theta_1)$$
$$L^* = L(\theta_2 | y) \propto k(y)P(y | \theta_2)$$
Uncertainty via the Likelihood Ratio

• **Compare two likelihood models**
  - unrestricted model: \( L^* \)
  - restricted (nested) model: \( L_R^* \)
  - Likelihood Ratio:

\[
L^* \geq L_R^* \implies \frac{L_R^*}{L^*} \leq 1
\]

• Likelihood ratio: the ratio of 2 traditional probabilities

\[
L_R^* = L(\theta_1|y) \propto k(y)P(y|\theta_1) \\
L^* = L(\theta_2|y) \propto k(y)P(y|\theta_2)
\]

\[
\frac{L(\theta_1|y)}{L(\theta_2|y)} = \frac{k(y)P(y|\theta_1)}{k(y)P(y|\theta_2)}
\]
Uncertainty via the Likelihood Ratio

- Compare **two likelihood models**
  - unrestricted model: \( L^* \)
  - restricted (nested) model: \( L_R^* \)
- Likelihood Ratio:

\[
L^* \geq L_R^* \implies \frac{L_R^*}{L^*} \leq 1
\]

- Likelihood ratio: the ratio of 2 traditional probabilities

\[
L_R^* = L(\theta_1|y) \propto k(y)P(y|\theta_1) \\
L^* = L(\theta_2|y) \propto k(y)P(y|\theta_2)
\]

\[
\frac{L(\theta_1|y)}{L(\theta_2|y)} = \frac{k(y)}{k(y)} \frac{P(y|\theta_1)}{P(y|\theta_2)} = \frac{P(y|\theta_1)}{P(y|\theta_2)}, \quad \text{a risk ratio}
\]
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L^*}{R} \right) = 2 \left( \ln L^* - \ln R \right) \]

\[ \sim \chi^2(r|m) \]

- \( r \) is the realized value of \( R \);
- \( m \) is the number of restricted parameters.

- If restrictions have no effect: \( E(R) = m \).

- Parameters are different from zero if: \( r \gg m \).

- Works well, but:
  - Lots of likelihood ratio tests may be required to test all points of interest.
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L^*}{R L^*} \right) = 2(\ln L^* - \ln R) \]

\( r \) is realized value of \( R \); \( m \) is number of restricted parameters

- If restrictions have no effect: \( E(R) = m \).
- Parameters are different from zero if: \( r \gg m \).
- Works well, but: Lots of likelihood ratio tests may be required to test all points of interest.
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L_R}{L^*} \right) \]
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L^*_R}{L^*} \right) = 2(\ln L^* - \ln L^*_R) \]
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L^*_R}{L^*} \right) = 2(\ln L^* - \ln L^*_R) \sim f_{\chi^2}(r|m) \]
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L_R^*}{L^*} \right) = 2(\ln L^* - \ln L_R^*) \sim f_{\chi^2}(r|m) \]

\( r \) is realized value of \( R \); \( m \) is number of restricted parameters
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[
R = -2 \ln \left( \frac{L_R^*}{L^*} \right) = 2(\ln L^* - \ln L_R^*) \sim f_{\chi^2}(r|m)
\]

\( r \) is realized value of \( R \); \( m \) is number of restricted parameters
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L_R^*}{L^*} \right) = 2(\ln L^* - \ln L_R^*) \sim f_{\chi^2}(r|m) \]

\( r \) is realized value of \( R \); \( m \) is number of restricted parameters

- If restrictions have no effect: \( E(R) = m \).
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L_R^*}{L^*} \right) = 2(\ln L^* - \ln L_R^*) \sim f_{\chi^2}(r|m) \]

\( r \) is realized value of \( R \); \( m \) is number of restricted parameters

- If restrictions have no effect: \( E(R) = m \).
- Parameters are different from zero if: \( r \gg m \)
Likelihood Ratio: Statistical Interpretation

Neyman-Pearson hypothesis testing (under the null):

\[ R = -2 \ln \left( \frac{L^*_R}{L^*_R} \right) = 2(\ln L^* - \ln L^*_R) \sim f_{\chi^2}(r|m) \]

\( r \) is realized value of \( R \); \( m \) is number of restricted parameters

- If restrictions have no effect: \( E(R) = m \).
- Parameters are different from zero if: \( r \gg m \)
- Works well, but: Lots of likelihood ratio tests may be required to test all points of interest
Uncertainty via Standard Errors

Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number. We use the normal likelihood to approximate all likelihoods (one justification: as $n \to \infty$, likelihoods become normal). Reformulate the normal (not stylized) likelihood with $E(Y) = \mu_i$:

$$L(\beta|y) \propto N(y_i|\mu_i, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y_i - \mu_i)^2}{2\sigma^2}\right)$$
Uncertainty via Standard Errors

• Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number.

• We use the normal likelihood to approximate all likelihoods (one justification: as \( n \to \infty \), likelihoods become normal).

• Reformulate the normal (not stylized) likelihood with \( E(Y) = \mu_i \):

\[
L(\beta|y) \propto N(y_i|\mu_i, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(y_i - \mu_i)^2}{2\sigma^2}\right)
\]

Uncertainty in Likelihood Inference
• Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number
• Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number

• We use the normal likelihood to approximate all likelihoods
• Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we **summarize the likelihood curvature near the maximum with one number**

• We use the normal likelihood to approximate all likelihoods

• (one justification: as $n \to \infty$, likelihoods become normal)
Uncertainty via Standard Errors

- Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number.
- We use the normal likelihood to approximate all likelihoods.
- (one justification: as $n \to \infty$, likelihoods become normal)
- Reformulate the normal (not stylized) likelihood with $E(Y) = \mu_i = \beta$: 

$$L(\theta | y) \propto N(y_i | \mu_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(y_i - \mu_i)^2}{2\sigma^2}\right)$$
Uncertainty via Standard Errors

Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number.

We use the normal likelihood to approximate all likelihoods.

(one justification: as \( n \to \infty \), likelihoods become normal)

Reformulate the normal (not stylized) likelihood with \( E(Y) = \mu_i = \beta \):

\[
L(\beta|y) \propto N(y_i|\mu_i, \sigma^2)
\]
• Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number

• We use the normal likelihood to approximate all likelihoods

• (one justification: as \( n \to \infty \), likelihoods become normal)

• Reformulate the normal (not stylized) likelihood with \( E(Y) = \mu_i = \beta \):

\[
L(\beta|y) \propto N(y_i|\mu_i, \sigma^2)
\]

\[
= (2\pi\sigma^2)^{-1/2} \exp\left(\frac{-(y_i - \mu_i)^2}{2\sigma^2}\right)
\]
Instead of (a) plotting the entire likelihood hyper-surface or (b) computing numerous likelihood ratio tests, we summarize the likelihood curvature near the maximum with one number.

We use the normal likelihood to approximate all likelihoods.

(one justification: as \( n \to \infty \), likelihoods become normal)

Reformulate the normal (not stylized) likelihood with \( E(Y) = \mu_i = \beta \):

\[
L(\beta|y) \propto N(y_i|\mu_i, \sigma^2)
\]

\[
= (2\pi \sigma^2)^{-1/2} \exp \left( -\frac{(y_i - \mu_i)^2}{2\sigma^2} \right)
\]

\[
= (2\pi \sigma^2)^{-1/2} \exp \left( -\frac{(y_i - \beta)^2}{2\sigma^2} \right)
\]
\[
\ln L(\beta|y) = -\frac{n}{2} \ln (2\pi \sigma^2) - \frac{1}{2\sigma^2} n \sum_{i=1}^{n} (y_i - \beta)^2
\]

\[
= -\frac{n}{2} \ln (2\pi \sigma^2) - \frac{1}{2\sigma^2} n \sum_{i=1}^{n} y_i^2 + 2y_i \beta - \beta^2
\]

\[
= a + b\beta + c\beta^2,
\]

A quadratic equation

• \( c = \left( -\frac{n}{2\sigma^2} \right) \) is the degree of curvature. Curvature is larger when:

• \( n \) is large
• \( \sigma^2 \) is small

• For normal likelihood, \( \left( -\frac{n}{2\sigma^2} \right) \) is a summary. The bigger the (negative) number…

• the better
• the more information exists in the MLE
• the larger the likelihood ratio would be in comparing the MLE with any other parameter value.
\[
\ln L(\beta|y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2
\]
\[
\ln L(\beta | y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2 \\
= -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i \beta + \beta^2)
\]
\[
\ln L(\beta|y) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2 \\
= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i\beta + \beta^2) \\
= \left( -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( -\frac{n}{2\sigma^2} \right) \beta^2
\]
\[ \ln L(\beta|y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2 \]

\[ = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i \beta + \beta^2) \]

\[ = \left( -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2 \]

\[ = a + b\beta + c\beta^2, \quad \text{A quadratic equation} \]
\( \ln L(\beta | y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2 \)

\[
= -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i\beta + \beta^2)
\]

\[
= \left( -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2
\]

\[
= a + b\beta + c\beta^2, \quad \text{A quadratic equation}
\]

- \( c = \left( \frac{-n}{2\sigma^2} \right) \) is the degree of curvature. Curvature is larger when:
\[ \ln L(\beta \mid y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2 \]

\[ = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i\beta + \beta^2) \]

\[ = \left( -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2 \]

\[ = a + b\beta + c\beta^2, \quad \text{A quadratic equation} \]

- \( c = \left( \frac{-n}{2\sigma^2} \right) \) is the degree of curvature. Curvature is larger when:
  - \( n \) is large
The likelihood function for the linear normal model is given by:
\[
\ln L(\beta|y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2
\]

Expanding this expression, we get:
\[
\ln L(\beta|y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i \beta + \beta^2)
\]

Simplifying further:
\[
\ln L(\beta|y) = \left( -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2
\]

\[
= a + b\beta + c\beta^2, \quad \text{A quadratic equation}
\]

- \(c = \left( \frac{-n}{2\sigma^2} \right)\) is the degree of curvature. Curvature is larger when:
  - \(n\) is large
  - \(\sigma^2\) is small
\[
\ln L(\beta|y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2
\]

\[
= -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i \beta + \beta^2)
\]

\[
= \left( -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2
\]

\[
= a + b\beta + c\beta^2,
\]

A quadratic equation

- \( c = \left( \frac{-n}{2\sigma^2} \right) \) is the degree of curvature. Curvature is larger when:
  - \( n \) is large
  - \( \sigma^2 \) is small
- For normal likelihood, \( \left( \frac{-n}{2\sigma^2} \right) \) is a summary. The bigger the (negative) number...
(Continued) Standard Errors, Linear Normal Model

\[
\ln L(\beta|y) = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2
\]

\[
= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i\beta + \beta^2)
\]

\[
= \left( -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2
\]

\[
= a + b\beta + c\beta^2, \quad \text{A quadratic equation}
\]

- \(c = \left( \frac{-n}{2\sigma^2} \right)\) is the degree of curvature. Curvature is larger when:
  - \(n\) is large
  - \(\sigma^2\) is small
- For normal likelihood, \(\left( \frac{-n}{2\sigma^2} \right)\) is a summary. The bigger the (negative) number...
  - the better
(Continued) Standard Errors, Linear Normal Model

\[
\ln L(\beta | y) = -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta)^2
\]

\[
= -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i\beta + \beta^2)
\]

\[
= \left( -\frac{n}{2} \ln(2\pi \sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right)\beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2
\]

\[
= a + b\beta + c\beta^2, \quad \text{A quadratic equation}
\]

- \( c = \left( \frac{-n}{2\sigma^2} \right) \) is the degree of curvature. Curvature is larger when:
  - \( n \) is large
  - \( \sigma^2 \) is small
- For normal likelihood, \( \left( \frac{-n}{2\sigma^2} \right) \) is a summary. The bigger the (negative) number...
  - the better
  - the more information exists in the MLE
ln \( L(\beta|y) \) = \(-\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n}(y_i - \beta)^2 \)

= \(-\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n}(y_i^2 - 2y_i\beta + \beta^2) \)

= \left( -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{\sum_{i=1}^{n} y_i^2}{2\sigma^2} \right) + \left( \frac{\sum_{i=1}^{n} y_i}{\sigma^2} \right) \beta + \left( \frac{-n}{2\sigma^2} \right) \beta^2

= a + b\beta + c\beta^2, \quad \text{A quadratic equation}

- \( c = \left( \frac{-n}{2\sigma^2} \right) \) is the degree of curvature. Curvature is larger when:
  - \( n \) is large
  - \( \sigma^2 \) is small

- For normal likelihood, \( \left( \frac{-n}{2\sigma^2} \right) \) is a summary. The bigger the (negative) number...
  - the better
  - the more information exists in the MLE
  - the larger the likelihood ratio would be in comparing the MLE with any other parameter value.
Standard Errors: Any Likelihood Model

When the log-likelihood is not normal, we'll use the best quadratic approximation to it. Under the normal,

$$\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = -n \sigma^2$$

Second derivative: coefficient $c$ on squared term for any model

We invert the curvature to provide a statistical interpretation:

$$\hat{V}(\hat{\theta}) = \left[-\frac{\partial^2 \ln L(\theta|y)}{\partial \theta \partial \theta'}\right]^{-1}_{\theta = \hat{\theta}}$$

The variance (aka covar or var-covar) across repeated samples

Quiz: How do we interpret $\hat{\sigma}_1$? What about $\hat{\sigma}_{21}$?

Works in general for a $k$-dimensional $\theta$ vector

Can be computed numerically

An estimate of a quadratic approximation to the log-likelihood

Asymptotically, it is an estimate of the exact log-likelihood
Standard Errors: Any Likelihood Model

- When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,
Standard Errors: Any Likelihood Model

- When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]
Standard Errors: Any Likelihood Model

- When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = \frac{-n}{\sigma^2}
\]

**Second derivative**: coefficient \( c \) on squared term for any model
Standard Errors: Any Likelihood Model

- When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]

**Second derivative:** coefficient \( c \) on squared term for any model

- We invert the curvature to provide a **statistical interpretation**: 
When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[ \frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = \frac{-n}{\sigma^2} \]

**Second derivative:** coefficient \( c \) on squared term for any model

We invert the curvature to provide a statistical interpretation:

\[ \hat{V}(\hat{\theta}) = \left[ -\frac{\partial^2 \ln L(\theta|y)}{\partial \theta \partial \theta'} \right]^{-1}_{\theta=\hat{\theta}} = \begin{pmatrix} \hat{\sigma}^2_1 & \hat{\sigma}_{12} & \cdots \\ \hat{\sigma}_{21} & \hat{\sigma}^2_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \]
Standard Errors: Any Likelihood Model

• When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]

Second derivative: coefficient \(c\) on squared term for any model

• We invert the curvature to provide a statistical interpretation:

\[
\hat{V}(\hat{\theta}) = \left[ -\frac{\partial^2 \ln L(\theta|y)}{\partial \theta \partial \theta'} \right]^{-1}_{\theta=\hat{\theta}} = \begin{pmatrix}
\hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots \\
\hat{\sigma}_{21} & \hat{\sigma}_2^2 & \cdots \\
\vdots & \vdots & \ddots 
\end{pmatrix}
\]

• The variance (aka covar or var-covar) across repeated samples
Standard Errors: Any Likelihood Model

- When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = \frac{-n}{\sigma^2}
\]

**Second derivative:** coefficient \( c \) on squared term for any model

- We invert the curvature to provide a statistical interpretation:

\[
\hat{V}(\hat{\theta}) = \left[ -\frac{\partial^2 \ln L(\theta|y)}{\partial \theta \partial \theta'} \right]^{-1}_{\theta = \hat{\theta}} = \begin{pmatrix}
\hat{\sigma}_1^2 & \hat{\sigma}_{12} & \ldots \\
\hat{\sigma}_{21} & \hat{\sigma}_2^2 & \ldots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

- The variance (aka covar or var-covar) across repeated samples
- **Quiz:** How do we interpret \( \hat{\sigma}_1 \)?
Standard Errors: Any Likelihood Model

• When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]

**Second derivative:** coefficient \( c \) on squared term for any model

• We invert the curvature to provide a statistical interpretation:

\[
\hat{V}(\hat{\theta}) = \left[-\frac{\partial^2 \ln L(\theta|y)}{\partial \theta \partial \theta'}\right]_{\theta=\hat{\theta}}^{-1} = \begin{pmatrix}
\hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots \\
\hat{\sigma}_{21} & \hat{\sigma}_2^2 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

• The variance (aka covar or var-covar) across repeated samples

• Quiz: How do we interpret \( \hat{\sigma}_1 \)? What about \( \hat{\sigma}_{21} \)?
Standard Errors: Any Likelihood Model

• When the log-likelihood is not normal, we’ll use the **best quadratic approximation** to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]

**Second derivative:** coefficient \( c \) on squared term for any model

• We invert the curvature to provide a **statistical interpretation**:

\[
\hat{V}(\hat{\theta}) = \left[-\frac{\partial^2 \ln L(\theta|y)}{\partial \theta \partial \theta'}\right]^{-1}_{\theta=\hat{\theta}} = \begin{pmatrix}
\hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots \\
\hat{\sigma}_{21} & \hat{\sigma}_2^2 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

• The variance (aka covar or var-covar) across repeated samples
• **Quiz:** How do we interpret \( \hat{\sigma}_1 \)? What about \( \hat{\sigma}_{21} \)?
• Works in general for a \( k \)-dimensional \( \theta \) vector
Standard Errors: Any Likelihood Model

• When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta | y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]

**Second derivative:** coefficient $c$ on squared term for any model

• We invert the curvature to provide a statistical interpretation:

\[
\hat{V}(\hat{\theta}) = \left[ -\frac{\partial^2 \ln L(\theta | y)}{\partial \theta \partial \theta'} \right]^{-1}_{\theta = \hat{\theta}} = \begin{pmatrix}
\hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots \\
\hat{\sigma}_{21} & \hat{\sigma}_2^2 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

• The variance (aka covar or var-covar) across repeated samples
• Quiz: How do we interpret $\hat{\sigma}_1$? What about $\hat{\sigma}_{21}$?
• Works in general for a $k$-dimensional $\theta$ vector
• Can be computed numerically
Standard Errors: Any Likelihood Model

- When the log-likelihood is not normal, we’ll use the best quadratic approximation to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta | y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]

Second derivative: coefficient \(c\) on squared term for any model
- We invert the curvature to provide a statistical interpretation:

\[
\hat{V}(\hat{\theta}) = \left[ -\frac{\partial^2 \ln L(\theta | y)}{\partial \theta \partial \theta'} \right]^{-1}_{\theta = \hat{\theta}} = \left( \begin{array}{ccc}
\hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots \\
\hat{\sigma}_{21} & \hat{\sigma}_2^2 & \cdots \\
\vdots & \vdots & \ddots
\end{array} \right)
\]

- The variance (aka covar or var-covar) across repeated samples
- Quiz: How do we interpret \(\hat{\sigma}_1\)? What about \(\hat{\sigma}_{21}\)?
- Works in general for a \(k\)-dimensional \(\theta\) vector
- Can be computed numerically
- An estimate of a quadratic approximation to the log-likelihood
Standard Errors: Any Likelihood Model

- When the log-likelihood is not normal, we’ll use the **best quadratic approximation** to it. Under the normal,

\[
\frac{\partial^2 \ln L(\beta|y)}{\partial \beta \partial \beta'} = -\frac{n}{\sigma^2}
\]

**Second derivative:** coefficient \( c \) on squared term for any model

- We invert the curvature to provide a **statistical interpretation**:

\[
\hat{V}(\hat{\theta}) = \left[-\frac{\partial^2 \ln L(\theta|y)}{\partial \theta \partial \theta'}\right]^{-1}_{\theta=\hat{\theta}} = \begin{pmatrix}
\hat{\sigma}_1^2 & \hat{\sigma}_{12} & \cdots \\
\hat{\sigma}_{21} & \hat{\sigma}_2^2 & \cdots \\
\vdots & \vdots & \ddots 
\end{pmatrix}
\]

- The variance (aka covar or var-covar) across repeated samples
- **Quiz:** How do we interpret \( \hat{\sigma}_1 \)? What about \( \hat{\sigma}_{21} \)?
- Works in general for a \( k \)-dimensional \( \theta \) vector
- Can be computed numerically

- **An estimate of a quadratic approximation** to the log-likelihood
- **Asymptotically**, it is an estimate of the exact log-likelihood
The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function
Parameter Simulation for *any* ML Model

- Assume model is correct (we'll come back to this!)
- Write down likelihood, calculate the MLE: $\hat{\theta}$
- Properties of $\hat{\theta}$ as $n$ gets large:
  - Distribution of $\hat{\theta}$ collapses to spike over $\theta$ (LLN $\Rightarrow$ consistency)
  - The standardized sampling distribution of $\hat{\theta}$ becomes normal (CLT $\Rightarrow$ asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves
- True variance of sampling distribution of $\hat{\theta}$: $V(\hat{\theta})$
- Estimate of $V(\hat{\theta})$: $\hat{V}(\hat{\theta})$, the inverse of the negative of the matrix of second derivatives of $\ln L(\theta|y)$, evaluated at $\hat{\theta}$.
- To simulate $\theta$,
  - Draw $\theta$ from the multivariate normal: $\tilde{\theta} \sim N(\hat{\theta}, \hat{V}(\hat{\theta}))$
  - This is an asymptotic approximation and can be wrong sometimes (we'll ignore now, improve later)

Quiz: What's the QOI? Is it $\theta$?
Parameter Simulation for *any* ML Model

- Assume model is correct
Parameter Simulation for *any* ML Model

- *Assume model is correct* (we’ll come back to this!)
Parameter Simulation for *any* ML Model

- Assume model is correct *(we’ll come back to this!)*
- Write down likelihood, calculate the MLE: $\hat{\theta}$

- Properties of $\hat{\theta}$ as $n$ gets large:
  - Distribution of $\hat{\theta}$ collapses to spike over $\theta$ *(LLN $\Rightarrow$ consistency)*
  - The standardized sampling distribution of $\hat{\theta}$ becomes normal *(CLT $\Rightarrow$ asymptotic normality)*
  - Quadratic approximation to the log-likelihood (from the second derivative) improves

- True variance of sampling distribution of $\hat{\theta}$:
- Estimate of $\text{Var}(\hat{\theta})$, the inverse of the negative of the matrix of second derivatives of $\ln L(\theta|y)$, evaluated at $\hat{\theta}$.

- To simulate $\theta$,
  - Draw $\theta$ from the multivariate normal: $\tilde{\theta} \sim N(\hat{\theta}, \hat{\text{Var}}(\hat{\theta}))$
  - This is an asymptotic approximation and can be wrong sometimes *(we’ll ignore now, improve later)*

- Quiz: What’s the QOI? Is it $\theta$?
Parameter Simulation for *any* ML Model

- Assume model is correct *(we’ll come back to this!)*
- Write down likelihood, calculate the MLE: $\hat{\theta}$
- Properties of $\hat{\theta}$ as $n$ gets large:
  - Distribution of $\hat{\theta}$ collapses to spike over $\theta$ (LLN $\Rightarrow$ consistency)
  - The standardized sampling distribution of $\hat{\theta}$ becomes normal (CLT $\Rightarrow$ asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves
  - True variance of sampling distribution of $\hat{\theta}$: $V(\hat{\theta})$
  - Estimate of $V(\hat{\theta})$: $\hat{V}(\hat{\theta})$, the inverse of the negative of the matrix of second derivatives of $\ln L(\theta|y)$, evaluated at $\hat{\theta}$.

To simulate $\theta$,
- Draw $\tilde{\theta}$ from the multivariate normal: $\tilde{\theta} \sim N(\hat{\theta}, \hat{V}(\hat{\theta}))$
- This is an asymptotic approximation and can be wrong sometimes (we’ll ignore now, improve later)

Quiz: What’s the QOI? Is it $\theta$?
Parameter Simulation for *any* ML Model

- Assume model is correct *(we’ll come back to this!)*
- Write down likelihood, calculate the MLE: $\hat{\theta}$
- Properties of $\hat{\theta}$ as $n$ gets large:
  - Distribution of $\hat{\theta}$ collapses to spike over $\theta$ (LLN $\Rightarrow$ consistency)
  - The standardized sampling distribution of $\hat{\theta}$ becomes normal (CLT $\Rightarrow$ asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves
  - True variance of sampling distribution of $\hat{\theta}$: $\mathbb{V}(\hat{\theta})$
  - Estimate of $\mathbb{V}(\hat{\theta})$: $\hat{\mathbb{V}}(\hat{\theta})$, the inverse of the negative of the matrix of second derivatives of $\ln L(\theta|y)$, evaluated at $\hat{\theta}$.

- To simulate $\theta$,
  - Draw $\tilde{\theta}$ from the multivariate normal: $\tilde{\theta} \sim \mathcal{N}(\hat{\theta}, \hat{\mathbb{V}}(\hat{\theta}))$
  - This is an asymptotic approximation and can be wrong sometimes (we’ll ignore now, improve later)
- Quiz: What’s the QOI? Is it $\theta$?
Parameter Simulation for *any* ML Model

- Assume model is correct *(we’ll come back to this!)*
- Write down likelihood, calculate the MLE: \( \hat{\theta} \)
- Properties of \( \hat{\theta} \) as \( n \) gets large:
  - Distribution of \( \hat{\theta} \) collapses to spike over \( \theta \) *(LLN \( \rightsquigarrow \) consistency)*
  - The standardized sampling distribution of \( \hat{\theta} \) becomes normal *(CLT \( \rightsquigarrow \) asymptotic normality)*
Parameter Simulation for *any* ML Model

- Assume model is correct *(we’ll come back to this!)*
- Write down likelihood, calculate the MLE: $\hat{\theta}$
- Properties of $\hat{\theta}$ as $n$ gets large:
  - Distribution of $\hat{\theta}$ collapses to spike over $\theta$ (LLN $\Rightarrow$ consistency)
  - The standardized sampling distribution of $\hat{\theta}$ becomes normal (CLT $\Rightarrow$ asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves

 Quiz: What’s the QOI? Is it $\theta$?
Parameter Simulation for *any* ML Model

- Assume model is correct (we’ll come back to this!)
- Write down likelihood, calculate the MLE: $\hat{\theta}$
- Properties of $\hat{\theta}$ as $n$ gets large:
  - Distribution of $\hat{\theta}$ collapses to spike over $\theta$ (LLN $\Rightarrow$ consistency)
  - The standardized sampling distribution of $\hat{\theta}$ becomes normal (CLT $\Rightarrow$ asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves
- True variance of sampling distribution of $\hat{\theta}$: $V(\hat{\theta})$
Parameter Simulation for any ML Model

- Assume model is correct (we’ll come back to this!)
- Write down likelihood, calculate the MLE: \( \hat{\theta} \)
- Properties of \( \hat{\theta} \) as \( n \) gets large:
  - Distribution of \( \hat{\theta} \) collapses to spike over \( \theta \) (LLN \( \Rightarrow \) consistency)
  - The standardized sampling distribution of \( \hat{\theta} \) becomes normal (CLT \( \Rightarrow \) asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves
- True variance of sampling distribution of \( \hat{\theta} \): \( V(\hat{\theta}) \)
- Estimate of \( V(\hat{\theta}) \): \( \hat{V}(\hat{\theta}) \), the inverse of the negative of the matrix of second derivatives of \( \ln L(\theta|y) \), evaluated at \( \hat{\theta} \).
Parameter Simulation for *any* ML Model

- Assume model is correct (we’ll come back to this!)
- Write down likelihood, calculate the MLE: \( \hat{\theta} \)
- Properties of \( \hat{\theta} \) as \( n \) gets large:
  - Distribution of \( \hat{\theta} \) collapses to spike over \( \theta \) (LLN \( \Rightarrow \) consistency)
  - The standardized sampling distribution of \( \hat{\theta} \) becomes normal (CLT \( \Rightarrow \) asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves
- True variance of sampling distribution of \( \hat{\theta} \): \( V(\hat{\theta}) \)
- Estimate of \( V(\hat{\theta}) \): \( \hat{V}(\hat{\theta}) \), the inverse of the negative of the matrix of second derivatives of \( \ln L(\theta|y) \), evaluated at \( \hat{\theta} \).
- To simulate \( \theta \),

Draw \( \tilde{\theta} \sim N(\hat{\theta}, \hat{V}(\hat{\theta})) \)

This is an asymptotic approximation and can be wrong sometimes (we’ll ignore now, improve later)

Quiz: What’s the QOI? Is it \( \theta \)?
Parameter Simulation for *any* ML Model

- **Assume model is correct** *(we’ll come back to this!)*
- **Write down likelihood, calculate the MLE:** \( \hat{\theta} \)
- **Properties of \( \hat{\theta} \) as \( n \) gets large:**
  - Distribution of \( \hat{\theta} \) collapses to spike over \( \theta \) (LLN \( \rightarrow \) consistency)
  - The standardized sampling distribution of \( \hat{\theta} \) becomes normal (CLT \( \rightarrow \) asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves

- **True variance of sampling distribution of \( \hat{\theta} \):** \( V(\hat{\theta}) \)
- **Estimate of \( V(\hat{\theta}) \):** \( \hat{V}(\hat{\theta}) \), the inverse of the negative of the matrix of second derivatives of \( \ln L(\theta|y) \), evaluated at \( \hat{\theta} \).
- **To simulate \( \theta \),**
  - **Draw \( \theta \) from the multivariate normal:** \( \tilde{\theta} \sim N(\hat{\theta}, \hat{V}(\hat{\theta})) \)
Parameter Simulation for *any* ML Model

- Assume model is correct (*we’ll* come back to this!)
- Write down likelihood, calculate the MLE: \( \hat{\theta} \)
- Properties of \( \hat{\theta} \) as \( n \) gets large:
  - Distribution of \( \hat{\theta} \) collapses to spike over \( \theta \) (LLN \( \rightsquigarrow \) consistency)
  - The standardized sampling distribution of \( \hat{\theta} \) becomes normal (CLT \( \rightsquigarrow \) asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves

- True variance of sampling distribution of \( \hat{\theta} \): \( V(\hat{\theta}) \)
- Estimate of \( V(\hat{\theta}) \): \( \hat{V}(\hat{\theta}) \), the inverse of the negative of the matrix of second derivatives of \( \ln L(\theta|y) \), evaluated at \( \hat{\theta} \).
- To simulate \( \theta \),
  - Draw \( \theta \) from the multivariate normal: \( \tilde{\theta} \sim N(\hat{\theta}, \hat{V}(\hat{\theta})) \)
  - This is an asymptotic approximation and can be wrong sometimes (*we’ll* ignore now, improve later)
Parameter Simulation for *any* ML Model

- **Assume model is correct** *(we’ll come back to this!)*
- **Write down likelihood, calculate the MLE**: $\hat{\theta}$
- **Properties of $\hat{\theta}$ as $n$ gets large**:
  - Distribution of $\hat{\theta}$ collapses to spike over $\theta$ (LLN $\Rightarrow$ consistency)
  - The standardized sampling distribution of $\hat{\theta}$ becomes normal (CLT $\Rightarrow$ asymptotic normality)
  - Quadratic approximation to the log-likelihood (from the second derivative) improves
- **True variance of sampling distribution of $\hat{\theta}$**: $V(\hat{\theta})$
- **Estimate of $V(\hat{\theta})$**: $\hat{V}(\hat{\theta})$, the inverse of the negative of the matrix of second derivatives of $\ln L(\theta|y)$, evaluated at $\hat{\theta}$.
- **To simulate $\theta$**,
  - Draw $\theta$ from the multivariate normal: $\tilde{\theta} \sim N(\hat{\theta}, \hat{V}(\hat{\theta}))$
  - This is an asymptotic approximation and can be wrong sometimes (we’ll ignore now, improve later)
- **Quiz**: What’s the QOI? Is it $\theta$?
QOI Simulation from *any* ML Model

Recall Generalized ML Model:

\[ Y_i \sim f(\theta_i, \alpha) \]

• stochastic

\[ \theta_i = g(x_i, \beta) \]

• systematic

Choose values of \( X \):

• Estimate:

\[ \hat{\gamma} = \{ \hat{\beta}, \hat{\alpha} \} \]

and its variance \( \hat{V}(\hat{\gamma}) \)

Simulate estimation uncertainty:

\[ \tilde{\gamma} \sim N[\hat{\gamma}, \hat{V}(\hat{\gamma})] \]

• Calculate (often expected value of \( y \)):

\[ \tilde{\theta}_c = g(X_c, \tilde{\beta}) \]

Simulate fundamental uncertainty:

\[ \tilde{y}_c \sim f(\tilde{\theta}_c, \tilde{\alpha}) \]

• Calculate QOI:

Calculate histogram, mean, variance, etc. of \( \tilde{y}_c \)
QOI Simulation from any ML Model
Overview here; Application to Linear Models Next; Then any QOI

• Recall Generalized ML Model:
  \[ Y_i \sim f(\theta_i, \alpha) \]
  \[ \theta_i = g(x_i, \beta) \]

• Choose values of \( X_c \):

• Estimate:
  \[ \hat{\gamma} = \{ \hat{\beta}, \hat{\alpha} \} \]
  and its variance \( \hat{V}(\hat{\gamma}) \)

• Simulate estimation uncertainty:
  \[ \tilde{\gamma} \sim N[\hat{\gamma}, \hat{V}(\hat{\gamma})] \]

• Calculate (often expected value of \( y \)):
  \[ \tilde{\theta}_c = g(X_c, \tilde{\beta}) \]

• Simulate fundamental uncertainty:
  \[ \tilde{y}_c \sim f(\tilde{\theta}_c, \tilde{\alpha}) \]

• Calculate QOI:
  Calculate histogram, mean, variance, etc. of \( \tilde{y}_c \)
QOI Simulation from \textit{any} ML Model

Overview here; Application to Linear Models Next; Then any QOI

- Recall Generalized ML Model:

\begin{align*}
Y_i & \sim f(\theta_i, \alpha) \quad \text{stochastic} \\
\theta_i & = g(x_i, \beta) \quad \text{systematic}
\end{align*}
QOI Simulation from *any* ML Model

Overview here; Application to Linear Models Next; Then any QOI

- **Recall Generalized ML Model:**

\[ Y_i \sim f(\theta_i, \alpha) \quad \text{stochastic} \]
\[ \theta_i = g(x_i, \beta) \quad \text{systematic} \]

- **Choose values of** \( X \): \( X_c \)
QOI Simulation from any ML Model
Overview here; Application to Linear Models Next; Then any QOI

• Recall Generalized ML Model:

\[
Y_i \sim f(\theta_i, \alpha) \quad \text{stochastic}
\]
\[
\theta_i = g(x_i, \beta) \quad \text{systematic}
\]

• Choose values of \( X \): \( X_c \)

• Estimate: MLE \( \hat{\gamma} = \{\hat{\beta}, \hat{\alpha}\} \) and its variance \( \hat{V}(\hat{\gamma}) \)
QOI Simulation from *any* ML Model

Overview here; Application to Linear Models Next; Then any QOI

- Recall Generalized ML Model:
  \[
  Y_i \sim f(\theta_i, \alpha) \quad \text{stochastic}
  \]
  \[
  \theta_i = g(x_i, \beta) \quad \text{systematic}
  \]
- Choose values of \( X \): \( X_c \)
- **Estimate**: MLE \( \hat{y} = \{\hat{\beta}, \hat{\alpha}\} \) and its variance \( \hat{V}(\hat{y}) \)
- **Simulate estimation uncertainty**: \( \tilde{y} \sim N[\hat{y}, \hat{V}(\hat{y})] \)
QOI Simulation from *any* ML Model

Overview here; Application to Linear Models Next; Then any QOI

- **Recall Generalized ML Model:**

  \[ Y_i \sim f(\theta_i, \alpha) \quad \text{stochastic} \]
  \[ \theta_i = g(x_i, \beta) \quad \text{systematic} \]

- **Choose values of** \(X\): \(X_c\)
- **Estimate:** MLE \(\hat{\gamma} = \{\hat{\beta}, \hat{\alpha}\}\) and its variance \(\hat{V}(\hat{\gamma})\)
- **Simulate estimation uncertainty:** \(\hat{\gamma} \sim N[\hat{\gamma}, \hat{V}(\hat{\gamma})]\)
- **Calculate** (often expected value of \(y\)): \(\tilde{\theta}_c = g(X_c, \tilde{\beta})\)
QOI Simulation from *any* ML Model

Overview here; Application to Linear Models Next; Then any QOI

- Recall Generalized ML Model:

  \[ Y_i \sim f(\theta_i, \alpha) \] stochastic

  \[ \theta_i = g(x_i, \beta) \] systematic

- Choose values of \( X \): \( X_c \)
- Estimate: MLE \( \hat{\gamma} = \{\hat{\beta}, \hat{\alpha}\} \) and its variance \( \hat{V}(\hat{\gamma}) \)
- Simulate estimation uncertainty: \( \tilde{\gamma} \sim N[\hat{\gamma}, \hat{V}(\hat{\gamma})] \)
- Calculate (often expected value of \( y \)): \( \tilde{\theta}_c = g(X_c, \tilde{\beta}) \)
- Simulate fundamental uncertainty: \( \tilde{y}_c \sim f(\tilde{\theta}_c, \tilde{\alpha}) \)
QOI Simulation from *any* ML Model

Overview here; Application to Linear Models Next; Then any QOI

- Recall Generalized ML Model:
  \[
  Y_i \sim f(\theta_i, \alpha) \quad \text{stochastic} \\
  \theta_i = g(x_i, \beta) \quad \text{systematic}
  \]

- Choose values of \( X \): \( X_c \)
- Estimate: MLE \( \hat{\gamma} = \{\hat{\beta}, \hat{\alpha}\} \) and its variance \( \hat{V}(\hat{\gamma}) \)
- Simulate estimation uncertainty: \( \tilde{\gamma} \sim N[\hat{\gamma}, \hat{V}(\hat{\gamma})] \)
- Calculate (often expected value of \( y \)): \( \tilde{\theta}_c = g(X_c, \tilde{\beta}) \)
- Simulate fundamental uncertainty: \( \tilde{y}_c \sim f(\tilde{\theta}_c, \tilde{\alpha}) \)
- Calculate QOI: Calculate histogram, mean, variance, etc. of \( \tilde{y}_c \)
Example: Forecasting Presidential Elections

The Data

\[ i \text{ U.S. state, for } i = 1, \ldots, 50 \]
\[ t \text{ election year, for } t = 1948, 1952, \ldots, 2016 \]
\[ y_{it} \text{ Democratic proportion of the two-party vote} \]
\[ X_{it} \text{ Constant, economics, polls, home state, ideology, etc.} \]
\[ X_i,2020 \text{ the same covariates as } X_{it} \text{ but measured in 2020} \]
\[ C_i \text{ The number of electoral college delegates in } i \text{ in 2020} \]

The Model

1. \[ Y_{it} \sim N(\mu_{it}, \sigma^2) \]
2. \[ \mu_{it} = x_{it} \beta \], where \( x_{it} \) includes a constant
3. \( Y_{it} \) and \( Y_{i' t'} \) are independent \( \forall i \neq i' \text{ and } t \neq t' \) (given \( X \))

Quiz: What are this model's weaknesses?
Example: Forecasting Presidential Elections

The Data

\[ Y_{it} \sim N(\mu_{it}, \sigma^2) \]

\[ \mu_{it} = x_{it} \beta, \] where \( x_{it} \) includes a constant

\[ Y_{it} \] and \( Y_{i't'} \) are independent \( \forall i \neq i' \) and \( t \neq t' \) (given \( X \))

Quiz: What are this model's weaknesses?
Example: Forecasting Presidential Elections

The Data

\[ i \quad \text{U.S. state, for } i = 1, \ldots, 50 \]
Example: Forecasting Presidential Elections

The Data

\[ \begin{align*} i & \text{ U.S. state, for } i = 1, \ldots, 50 \\ t & \text{ election year, for } t = 1948, 1952, \ldots, 2016 \end{align*} \]
Example: Forecasting Presidential Elections

The Data

\[ i \quad \text{U.S. state, for } i = 1, \ldots, 50 \]
\[ t \quad \text{election year, for } t = 1948, 1952, \ldots, 2016 \]
\[ y_{it} \quad \text{Democratic proportion of the two-party vote} \]
Example: Forecasting Presidential Elections

The Data

\begin{align*}
i & \quad \text{U.S. state, for } i = 1, \ldots, 50 \\
t & \quad \text{election year, for } t = 1948, 1952, \ldots, 2016 \\
y_{it} & \quad \text{Democratic proportion of the two-party vote} \\
X_{it} & \quad \text{Constant, economics, polls, home state, ideology, etc.}
\end{align*}
Example: Forecasting Presidential Elections

The Data

\(i\) U.S. state, for \(i = 1, \ldots, 50\)
\(t\) election year, for \(t = 1948, 1952, \ldots, 2016\)
\(y_{it}\) Democratic proportion of the two-party vote
\(X_{it}\) Constant, economics, polls, home state, ideology, etc.
\(X_{i,2020}\) the same covariates as \(X_{it}\) but measured in 2020

The Model

1. \(Y_{it} \sim N(\mu_{it}, \sigma^2)\)
2. \(\mu_{it} = x_{it} \beta\), where \(x_{it}\) includes a constant
3. \(Y_{it}\) and \(Y_{i't'}\) are independent \(\forall i \neq i'\) and \(t \neq t'\) (given \(X\))
The Data

- **i** U.S. state, for \( i = 1, \ldots, 50 \)
- **t** election year, for \( t = 1948, 1952, \ldots, 2016 \)
- \( y_{it} \) Democratic proportion of the two-party vote
- \( X_{it} \) Constant, economics, polls, home state, ideology, etc.
- \( X_{i,2020} \) the same covariates as \( X_{it} \) but measured in 2020
- \( C_i \) The number of electoral College delegates in \( i \) in 2020
Example: Forecasting Presidential Elections

The Data

\begin{align*}
  i & \text{ U.S. state, for } i = 1, \ldots, 50 \\
  t & \text{ election year, for } t = 1948, 1952, \ldots, 2016 \\
  y_{it} & \text{ Democratic proportion of the two-party vote} \\
  X_{it} & \text{ Constant, economics, polls, home state, ideology, etc.} \\
  X_{i,2020} & \text{ the same covariates as } X_{it} \text{ but measured in 2020} \\
  C_i & \text{ The number of electoral College delegates in } i \text{ in 2020}
\end{align*}

The Model

1. \[ Y_{it} \sim \mathcal{N}(\mu_{it}, \sigma^2) \]
2. \[ \mu_{it} = x_{it} \beta \], where \( x_{it} \) includes a constant
3. \( Y_{it} \) and \( Y_{i',t'} \) are independent \( \forall i \neq i' \) and \( t \neq t' \) (given \( X \))

Quiz: What are this model's weaknesses?
Example: Forecasting Presidential Elections

The Data

- $i$: U.S. state, for $i = 1, \ldots, 50$
- $t$: election year, for $t = 1948, 1952, \ldots, 2016$
- $y_{it}$: Democratic proportion of the two-party vote
- $X_{it}$: Constant, economics, polls, home state, ideology, etc.
- $X_{i,2020}$: the same covariates as $X_{it}$ but measured in 2020
- $C_i$: The number of electoral College delegates in $i$ in 2020

The Model

1. $Y_{it} \sim N(\mu_{it}, \sigma^2)$. 

Example: Forecasting Presidential Elections

The Data

\( i \) 
U.S. state, for \( i = 1, \ldots, 50 \)

\( t \) 
election year, for \( t = 1948, 1952, \ldots, 2016 \)

\( y_{it} \) 
Democratic proportion of the two-party vote

\( X_{it} \) 
Constant, economics, polls, home state, ideology, etc.

\( X_{i,2020} \) 
the same covariates as \( X_{it} \) but measured in 2020

\( C_i \) 
The number of electoral College delegates in \( i \) in 2020

The Model

1. \( Y_{it} \sim N(\mu_{it}, \sigma^2) \).

2. \( \mu_{it} = x_{it} \beta \), where \( x_{it} \) includes a constant

Quiz: What are this model's weaknesses?

Simulation from Likelihood Models
Example: Forecasting Presidential Elections

The Data

\[ i \quad \text{U.S. state, for } i = 1, \ldots, 50 \]
\[ t \quad \text{election year, for } t = 1948, 1952, \ldots, 2016 \]
\[ y_{it} \quad \text{Democratic proportion of the two-party vote} \]
\[ X_{it} \quad \text{Constant, economics, polls, home state, ideology, etc.} \]
\[ X_{i,2020} \quad \text{the same covariates as } X_{it} \text{ but measured in 2020} \]
\[ C_i \quad \text{The number of electoral College delegates in } i \text{ in 2020} \]

The Model

1. \[ Y_{it} \sim N(\mu_{it}, \sigma^2). \]
2. \[ \mu_{it} = x_{it}\beta, \text{ where } x_{it} \text{ includes a constant} \]
3. \[ Y_{it} \text{ and } Y_{i't'} \text{ are independent } \forall i \neq i' \text{ and } t \neq t' \text{ (given } X) \]
Example: Forecasting Presidential Elections

The Data

\( i \) U.S. state, for \( i = 1, \ldots, 50 \)

\( t \) election year, for \( t = 1948, 1952, \ldots, 2016 \)

\( y_{it} \) Democratic proportion of the two-party vote

\( X_{it} \) Constant, economics, polls, home state, ideology, etc.

\( X_{i,2020} \) the same covariates as \( X_{it} \) but measured in 2020

\( C_i \) The number of electoral College delegates in \( i \) in 2020

The Model

1. \( Y_{it} \sim N(\mu_{it}, \sigma^2). \)
2. \( \mu_{it} = x_{it}\beta, \) where \( x_{it} \) includes a constant
3. \( Y_{it} \) and \( Y_{i',t'} \) are independent \( \forall \ i \neq i' \) and \( t \neq t' \) (given \( X \))

Quiz: What are this model’s weaknesses?
The Likelihood Model

• Likelihood for observation

\[ L(\mu_{it}, \sigma^2 | y_{it}) \propto N(y_{it} | \mu_{it}, \sigma^2) \]

= \left( \frac{2\pi\sigma^2}{42.36} \right)^{-1/2} e^{-\frac{(y_{it} - \mu_{it})^2}{42.36}}

• Likelihood for all \( n \) observations

\[ L(\beta, \sigma^2 | y) = n \prod_{i=1}^n T \prod_{t=1}^T L(\mu_{it}, \sigma^2 | y_{it}) \]

= \left( \frac{2\pi\sigma^2}{41.61} \right)^{-1/2} e^{-\frac{(y_{it} - \mu_{it})^2}{41.61}}

Simulation from Likelihood Models
The Likelihood Model

- Likelihood for observation $i$.

\[ L(\mu_{it}, \sigma^2|y_{it}) \propto N(y_{it}|\mu_{it}, \sigma^2) \]
The Likelihood Model

- Likelihood for observation $i$th

$$L(\mu_{it}, \sigma^2 \mid y_{it}) \propto N(y_{it} \mid \mu_{it}, \sigma^2) = (2\pi \sigma^2)^{-1/2} e^{-\frac{(y_{it} - \mu_{it})^2}{2\sigma^2}}$$
The Likelihood Model

- Likelihood for observation \( it \)

\[
L(\mu_{it}, \sigma^2 | y_{it}) \propto N(y_{it} | \mu_{it}, \sigma^2) = (2\pi \sigma^2)^{-1/2} e^{-\frac{(y_{it} - \mu_{it})^2}{2\sigma^2}}
\]

- Likelihood for all \( n \) observations

\[
L(\beta, \sigma^2 | y) = \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it})
\]
The Likelihood Model

- Likelihood for observation \( it \)
  \[
  L(\mu_{it}, \sigma^2|y_{it}) \propto N(y_{it}|\mu_{it}, \sigma^2) = (2\pi\sigma^2)^{-1/2} e^{-\frac{(y_{it}-\mu_{it})^2}{2\sigma^2}}
  \]

- Likelihood for all \( n \) observations
  \[
  L(\beta, \sigma^2|y) = \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2|y_{it})
  = \prod_{i=1}^{n} \prod_{t=1}^{T} (2\pi\sigma^2)^{-1/2} e^{-\frac{(y_{it}-\mu_{it})^2}{2\sigma^2}}
  \]
Log-Likelihood

\[
\ln L(\beta, \sigma^2 | y) = \ln \left[ n \prod_{i=1}^{N} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it}) \right] = \sum_{i=1}^{N} \sum_{t=1}^{T} \ln L(y_{it} | \mu_{it}, \sigma^2) = \sum_{i=1}^{N} \sum_{t=1}^{T} \ln \left[ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_{it} - \mu_{it})^2}{2\sigma^2}} \right] = -\frac{1}{2} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ \ln(2\pi\sigma^2) + \frac{(y_{it} - \mu_{it})^2}{\sigma^2} \right] = -\frac{1}{2} \sum_{i=1}^{N} \sum_{t=1}^{T} \left[ \ln(2\pi\sigma^2) + \frac{(y_{it} - X_{it}^T \beta)^2}{\sigma^2} \right]
\]
Log-Likelihood

\[
\ln L(\beta, \sigma^2 | y) = \ln \left[ \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it}) \right]
\]
Log-Likelihood

\[
\begin{align*}
\ln L(\beta, \sigma^2 | y) &= \ln \left[ \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it}) \right] \\
&= \sum_{i=1}^{n} \sum_{t=1}^{T} \ln L(y_{it} | \mu_{it}, \sigma^2)
\end{align*}
\]
Log-Likelihood

\[
\ln L(\beta, \sigma^2|y) = \ln \left[ \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2|y_{it}) \right] = \sum_{i=1}^{n} \sum_{t=1}^{T} \ln L(y_{it}|\mu_{it}, \sigma^2)
\]

\[
= \sum_{i=1}^{n} \sum_{t=1}^{T} \ln \left[ (2\pi \sigma^2)^{-1/2} e^{-\frac{(y_{it}-\mu_{it})^2}{2\sigma^2}} \right]
\]
Log-Likelihood

$$\ln L(\beta, \sigma^2 | y) = \ln \left[ \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it}) \right] = \sum_{i=1}^{n} \sum_{t=1}^{T} \ln L(y_{it} | \mu_{it}, \sigma^2)$$

$$= \sum_{i=1}^{n} \sum_{t=1}^{T} \ln \left[ (2\pi \sigma^2)^{-1/2} e^{-\frac{(y_{it}-\mu_{it})^2}{2\sigma^2}} \right]$$

$$= \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ -\frac{1}{2} \ln(2\pi \sigma^2) - \frac{(y_{it} - \mu_{it})^2}{2\sigma^2} \right]$$
Log-Likelihood

\[
\ln L(\beta, \sigma^2 | y) = \ln \left[ \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it}) \right] = \sum_{i=1}^{n} \sum_{t=1}^{T} \ln L(y_{it} | \mu_{it}, \sigma^2)
\]

\[
= \sum_{i=1}^{n} \sum_{t=1}^{T} \ln \left[ (2\pi \sigma^2)^{-1/2} e^{-\frac{(y_{it} - \mu_{it})^2}{2\sigma^2}} \right]
\]

\[
= \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ -\frac{1}{2} \ln(2\pi \sigma^2) - \frac{(y_{it} - \mu_{it})^2}{2\sigma^2} \right]
\]

\[
= -\frac{1}{2} \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ \ln(2\pi) + \ln \sigma^2 + \frac{(y_{it} - \mu_{it})^2}{\sigma^2} \right]
\]
Log-Likelihood

\[
\ln L(\beta, \sigma^2 | y) = \ln \left[ \prod_{i=1}^{n} \prod_{t=1}^{T} L(\mu_{it}, \sigma^2 | y_{it}) \right] = \sum_{i=1}^{n} \sum_{t=1}^{T} \ln L(y_{it} | \mu_{it}, \sigma^2)
\]

\[
= \sum_{i=1}^{n} \sum_{t=1}^{T} \ln \left[ (2\pi \sigma^2)^{-1/2} e^{-\frac{(y_{it} - \mu_{it})^2}{2\sigma^2}} \right]
\]

\[
= \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ -\frac{1}{2} \ln(2\pi \sigma^2) - \frac{(y_{it} - \mu_{it})^2}{2\sigma^2} \right]
\]

\[
= -\frac{1}{2} \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ \ln(2\pi) + \ln \sigma^2 + \frac{(y_{it} - \mu_{it})^2}{\sigma^2} \right]
\]

\[
= -\frac{1}{2} \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ \ln \sigma^2 + \frac{(y_{it} - \mu_{it})^2}{\sigma^2} \right]
\]
Log-Likelihood

\[
\ln L(\beta, \sigma^2 | y) = \ln \left[ \prod_{i=1}^n \prod_{t=1}^T L(\mu_{it}, \sigma^2 | y_{it}) \right] = \sum_{i=1}^n \sum_{t=1}^T \ln L(y_{it} | \mu_{it}, \sigma^2)
\]

\[
= \sum_{i=1}^n \sum_{t=1}^T \ln \left[ (2\pi \sigma^2)^{-1/2} e^{-\frac{(y_{it} - \mu_{it})^2}{2\sigma^2}} \right]
\]

\[
= \sum_{i=1}^n \sum_{t=1}^T \left[ -\frac{1}{2} \ln(2\pi \sigma^2) - \frac{(y_{it} - \mu_{it})^2}{2\sigma^2} \right]
\]

\[
= -\frac{1}{2} \sum_{i=1}^n \sum_{t=1}^T \left[ \ln(2\pi) + \ln \sigma^2 + \frac{(y_{it} - \mu_{it})^2}{\sigma^2} \right]
\]

\[
= -\frac{1}{2} \sum_{i=1}^n \sum_{t=1}^T \left[ \ln \sigma^2 + \frac{(y_{it} - \mu_{it})^2}{\sigma^2} \right]
\]

\[
= -\frac{1}{2} \sum_{i=1}^n \sum_{t=1}^T \left[ \ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right]
\]
Estimation

• Reparameterize to unbounded scale
  • Numerical optimizers work better this way
  • The CLT kicks in faster
  • $\beta$ is already unbounded
  • $\sigma > 0 \Rightarrow$ transform with $\sigma = e^{\eta}$, and estimate $\eta$

• Stack: $\gamma = \{\beta, \eta\}$, a $k + 2 \times 1$ vector ($k$: number of covariates)

• Turn log-likelihood into code; maximize so we can get:
  • Point estimates: save the MLE, $\hat{\gamma} = \{\hat{\beta}, \hat{\eta}\}$
  • Uncertainty estimates: $\hat{V}(\hat{\gamma})$, which is $k + 2 \times k + 2$
Reparameterize to unbounded scale
Estimation

- Reparameterize to unbounded scale
  - numerical optimizers work better this way
Estimation

- **Reparameterize to unbounded scale**
  - numerical optimizers work better this way
  - the CLT kicks in faster

\[
\beta \text{ is already unbounded} \Rightarrow \text{transform with } \sigma = e^{\eta}, \text{ and estimate } \eta
\]

- Stack: \( \gamma = \{\beta, \eta\} \), a \( k + 2 \times 1 \) vector (\( k \): number of covariates)

- Turn log-likelihood into code; maximize so we can get:
  - Point estimates: save the MLE, \( \hat{\gamma} = \{\hat{\beta}, \hat{\eta}\} \)
  - Uncertainty estimates: \( \hat{V}(\hat{\gamma}) \), which is \( k + 2 \times k + 2 \)
Estimation

- **Reparameterize to unbounded scale**
  - numerical optimizers work better this way
  - the CLT kicks in faster
  - $\beta$ is already unbounded
Estimation

• Reparameterize to unbounded scale
  • numerical optimizers work better this way
  • the CLT kicks in faster
  • $\beta$ is already unbounded
  • $\sigma > 0 \sim$ transform with $\sigma = e^\eta$, and estimate $\eta$
Estimation

- Reparameterize to unbounded scale
  - numerical optimizers work better this way
  - the CLT kicks in faster
  - $\beta$ is already unbounded
  - $\sigma > 0 \implies$ transform with $\sigma = e^{\eta}$, and estimate $\eta$

- Stack: $\gamma = \{\beta, \eta\}$, a $k + 2 \times 1$ vector ($k$: number of covariates)
Estimation

• Reparameterize to unbounded scale
  • numerical optimizers work better this way
  • the CLT kicks in faster
  • $\beta$ is already unbounded
  • $\sigma > 0 \Rightarrow$ transform with $\sigma = e^\eta$, and estimate $\eta$

• Stack: $\gamma = \{\beta, \eta\}$, a $k + 2 \times 1$ vector ($k$: number of covariates)

• Turn log-likelihood into code; maximize so we can get:
Estimation

- Reparameterize to unbounded scale
  - numerical optimizers work better this way
  - the CLT kicks in faster
  - $\beta$ is already unbounded
  - $\sigma > 0 \implies$ transform with $\sigma = e^\eta$, and estimate $\eta$

- Stack: $\gamma = \{\beta, \eta\}$, a $k + 2 \times 1$ vector ($k$: number of covariates)

- Turn log-likelihood into code; maximize so we can get:
  - Point estimates: save the MLE, $\hat{\gamma} = \{\hat{\beta}, \hat{\eta}\}$
Estimation

- **Reparameterize to unbounded scale**
  - numerical optimizers work better this way
  - the CLT kicks in faster
  - $\beta$ is already unbounded
  - $\sigma > 0 \iff$ transform with $\sigma = e^{\eta}$, and estimate $\eta$

- **Stack:** $\gamma = \{\beta, \eta\}$, a $k + 2 \times 1$ vector ($k$: number of covariates)

- **Turn log-likelihood into code; maximize so we can get:**
  - **Point estimates:** save the MLE, $\hat{\gamma} = \{\hat{\beta}, \hat{\eta}\}$
  - **Uncertainty estimates:** $\hat{V}(\hat{\gamma})$, which is $k + 2 \times k + 2$
R Code for the Log-Likelihood

• (Recall) mathematical Form:

\[
\ln L(\beta, \sigma^2 | y) = n \sum_{i=1}^T \sum_{t=1}^T -1/2 \left[ \ln \sigma^2 + (y_{it} - X_{it} \beta)^2 / \sigma^2 \right]
\]

• An R function:

```r
loglik <- function(par, X, Y) {
  X <- as.matrix(cbind(1, X))
  beta <- par[1:ncol(X)]
  sigma2 <- exp(par[ncol(X) + 1])^-1/2 * sum(log(sigma2) + ((Y - X %*% beta)^2) / sigma2)
}
```

• Calling it:

```r
loglik(c(2,1,2,1,33,4,2), x, y)
loglik(c(2,1,2,1,33,4,7), x, y)
loglik(c(2,1,2,1,33,4,5), x, y)
```
R Code for the Log-Likelihood

• (Recall) mathematical Form:

\[
\ln L(\beta, \sigma^2|y) = \sum_{i=1}^{n} \sum_{t=1}^{T} -\frac{1}{2} \left[ \ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right]
\]
R Code for the Log-Likelihood

- (Recall) mathematical Form:

\[
\ln L(\beta, \sigma^2 | y) = \sum_{i=1}^{n} \sum_{t=1}^{T} - \frac{1}{2} \left[ \ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right]
\]

- An R function:

```r
code
loglik <- function(par, X, Y) {
  X <- as.matrix(cbind(1, X))
  beta <- par[1:ncol(X)]
  sigma2 <- exp(par[ncol(X) + 1])
  -1/2*sum(log(sigma2) + ((Y - X %*% beta)^2)/sigma2)
}
```

Simulation from Likelihood Models
R Code for the Log-Likelihood

- (Recall) mathematical Form:

\[ \ln L(\beta, \sigma^2|y) = \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ -\frac{1}{2} \ln \sigma^2 + \frac{(y_{it} - X_{it}\beta)^2}{\sigma^2} \right] \]

- An R function:

```r
loglik <- function(par, X, Y) {
  X <- as.matrix(cbind(1, X))
  beta <- par[1:ncol(X)]
  sigma2 <- exp(par[ncol(X) + 1])
  -1/2*sum(log(sigma2) + ((Y - X %*% beta)^2)/sigma2)
}
```

- Calling it:

```r
loglik(c(2,1,2,1,33,4,2),x,y)
loglik(c(2,1,2,1,33,4,7),x,y)
loglik(c(2,1,2,1,33,4,5),x,y)
```
Quantities of Interest in this election data set

- Quiz: What are the QOIs?
  - There's no right answer; here's mine:
  - (Reasons we care about the regression coefficients: No)
  - Predictive distribution of Dem electoral college delegates
  - Expected number of Dem electoral college delegates
  - Probability that Dem candidate is elected: $\frac{1}{n} \sum_{i=1}^{n} C_i > 0.5$

Simulation from Likelihood Models
Quantities of Interest in this election data set

• Quiz: What are the QOIs?
Quantities of Interest in this election data set

- **Quiz: What are the QOIs?**
- **There’s no right answer; here’s mine:**
Quantities of Interest in this election data set

- **Quiz: What are the QOIs?**
- **There’s no right answer; here’s mine:**
  - (Reasons we care about the regression coefficients: )
Quantities of Interest in this election data set

- **Quiz:** What are the QOIs?
- There’s no right answer; here’s mine:
  - (Reasons we care about the regression coefficients: N )
Quantities of Interest in this election data set

- **Quiz:** What are the QOIs?
- There’s no right answer; here’s mine:
  - (Reasons we care about the regression coefficients: No)
Quantities of Interest in this election data set

- Quiz: What are the QOIs?
- There’s no right answer; here’s mine:
  - (Reasons we care about the regression coefficients: Non )
Quantities of Interest in this election data set

- Quiz: What are the QOIs?
- There’s no right answer; here’s mine:
  - (Reasons we care about the regression coefficients: None)

Simulation from Likelihood Models 46/53
Quantities of Interest in this election data set

• Quiz: What are the QOIs?
• There’s no right answer; here’s mine:
  • (Reasons we care about the regression coefficients: None)
  • Predictive distribution of Dem electoral college delegates
Quantities of Interest in this election data set

- **Quiz:** What are the QOIs?
- There’s no right answer; here’s mine:
  - (Reasons we care about the regression coefficients: None)
  - Predictive **distribution** of Dem electoral college delegates
  - **Expected number** of Dem electoral college delegates
Quantities of Interest in this election data set

- **Quiz:** What are the QOIs?
- **There’s no right answer; here’s mine:**
  - (Reasons we care about the regression coefficients: None)
  - Predictive distribution of Dem electoral college delegates
  - Expected number of Dem electoral college delegates
  - Probability that Dem candidate is elected: gets more than \( \sum_{i=1}^{n} C_i/n > 0.5 \) proportion of electoral college delegates
Predicting Allocations of Electoral College Delegates

Quiz: how to simulate predictions of $C_i$ in state $i$?

Options:
1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$

Quiz: What's your prediction if $\hat{y}_{i,2020} = 0.51 \forall i$?

Problem: ignores fundamental uncertainty

2. Allocate $C_i \hat{y}_{i,2020}$ to Dems; $C_i (1 - \hat{y}_{i,2020})$ to Reps

Quiz: What happens if $\hat{y}_{i,2020}$ is uncertain?

Problem: ignores estimation uncertainty

Quiz: How might we also include estimation uncertainty?
Predicting Allocations of Electoral College Delegates

- Quiz: how to simulate predictions of $C_i$ in state $i$?
Predicting Allocations of Electoral College Delegates

- Quiz: how to simulate predictions of $C_i$ in state $i$?
- Options:
Predicting Allocations of Electoral College Delegates

- **Quiz:** how to simulate predictions of $C_i$ in state $i$?
- **Options:**
  1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$
• Quiz: how to simulate predictions of $C_i$ in state $i$?

• Options:
  1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$
  
  • Quiz: What’s your prediction if $\hat{y}_{i,2020} = 0.51 \forall i$?
Predicting Allocations of Electoral College Delegates

- **Quiz:** how to simulate predictions of $C_i$ in state $i$?
  - **Options:**
    1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$
    
    - **Quiz:** What’s your prediction if $\hat{y}_{i,2020} = 0.51 \ \forall \ i$?
    - **Problem:** ignores fundamental uncertainty
Predicting Allocations of Electoral College Delegates

- **Quiz:** how to simulate predictions of $C_i$ in state $i$?
- **Options:**
  1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$
     - **Quiz:** What’s your prediction if $\hat{y}_{i,2020} = 0.51 \forall i$?
     - **Problem:** ignores fundamental uncertainty
  2. Allocate $C_i \hat{y}_{i,2020}$ to Dems; $C_i (1 - \hat{y}_{i,2020})$ to Reps

Simulation from Likelihood Models
Quiz: how to simulate predictions of $C_i$ in state $i$?

Options:

1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$
   - Quiz: What’s your prediction if $\hat{y}_{i,2020} = 0.51 \ \forall \ i$?
   - Problem: ignores fundamental uncertainty

2. Allocate $C_i \hat{y}_{i,2020}$ to Dems; $C_i(1 - \hat{y}_{i,2020})$ to Reps
   - Quiz: What happens if $\hat{y}_{i,2020}$ is uncertain?
Predicting Allocations of Electoral College Delegates

- **Quiz:** how to simulate predictions of $C_i$ in state $i$?
- **Options:**
  1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$
     - **Quiz:** What’s your prediction if $\hat{y}_{i,2020} = 0.51 \forall i$?
     - **Problem:** ignores fundamental uncertainty
  2. Allocate $C_i\hat{y}_{i,2020}$ to Dems; $C_i(1 - \hat{y}_{i,2020})$ to Reps
     - **Quiz:** What happens if $\hat{y}_{i,2020}$ is uncertain?
     - **Problem:** Ignores estimation uncertainty
Quiz: how to simulate predictions of $C_i$ in state $i$?

Options:

1. if $\hat{y}_{i,2020} > 0.5$, Dems get all $C_i$; otherwise, Reps get all $C_i$
   - Quiz: What’s your prediction if $\hat{y}_{i,2020} = 0.51 \forall i$?
   - Problem: ignores fundamental uncertainty

2. Allocate $C_i \hat{y}_{i,2020}$ to Dems; $C_i (1 - \hat{y}_{i,2020})$ to Reps
   - Quiz: What happens if $\hat{y}_{i,2020}$ is uncertain?
   - Problem: Ignores estimation uncertainty
   - Quiz: How might we also include estimation uncertainty?
Predictive Distribution of Electoral College Delegates
Including fundamental and estimation uncertainty

Simulate 1,000 national elections (⇒ number of Dem delegates)

• For state $i$ (repeat for $i = 1, \ldots, 51$)

1. Draw $\tilde{y}_{i,2020}$ from its distribution for state $i$,
   $\tilde{y}_{i,2020} \sim P(y_{i,2020} | y_{i,t}, t < 2020; X_{i,t}', t \leq 2020)$
   i.e. $P(\text{unknown} | \text{data})$. (Details shortly.)

2. If $\tilde{y}_{i,2020} > 0.5$ Dems "win" $C_i$ electoral college delegates (Reps get 0); otherwise, Dems get 0 (Reps get $C_i$)

• Calculate total Dem delegates nationally: add simulated winnings from all states:
  $\sum_{i=1}^{51} 1(\tilde{y}_{i,2020} > 0.5)C_i$

• Calculate QOIs: average, standard deviation, histogram
Predictive Distribution of Electoral College Delegates
Including fundamental and estimation uncertainty

- Simulate 1,000 national elections (\(\sim\) number of Dem delegates)
Predictive Distribution of Electoral College Delegates
Including fundamental and estimation uncertainty

- Simulate 1,000 national elections (∼ number of Dem delegates)
- Calculate QOIs: average, standard deviation, histogram
Predictive Distribution of Electoral College Delegates
Including fundamental and estimation uncertainty

• Simulate 1,000 national elections (\(\sim\) number of Dem delegates)

  • For state \(i\) (repeat for \(i = 1, \ldots, 51\))

• Calculate QOIs: average, standard deviation, histogram
Predictive Distribution of Electoral College Delegates
Including fundamental and estimation uncertainty

• Simulate 1,000 national elections (\(\sim\) number of Dem delegates)

• For state \(i\) (repeat for \(i = 1, \ldots, 51\))
  1. Draw \(y_{i,2020}\) from its distribution for state \(i\),

\[
\tilde{y}_{i,2020} \sim P(y_{i,2020}|y_{it}, t < 2020; X_{it'}, t' \leq 2020)
\]

  i.e. \(P(\text{unknown}|\text{data})\). (Details shortly.)

• Calculate QOIs: average, standard deviation, histogram
Predictive Distribution of Electoral College Delegates
Including fundamental and estimation uncertainty

- Simulate 1,000 national elections (∼ number of Dem delegates)
  
  - For state $i$ (repeat for $i = 1, \ldots, 51$)
    1. Draw $y_{i,2020}$ from its distribution for state $i$,

\[
\tilde{y}_{i,2020} \sim P(y_{i,2020} | y_{it}, t < 2020; X_{it'}, t' \leq 2020)
\]

  i.e. $P($unknown$|data)$. (Details shortly.)

  2. If $\tilde{y}_{i,2020} > 0.5$ Dems “win” $C_i$ electoral college delegates (Reps get 0); otherwise, Dems get 0 (Reps get $C_i$)

- Calculate QOIs: average, standard deviation, histogram
Predictive Distribution of Electoral College Delegates
Including fundamental and estimation uncertainty

- Simulate 1,000 national elections (∼ number of Dem delegates)
  - For state $i$ (repeat for $i = 1, \ldots, 51$)
    1. Draw $y_{i,2020}$ from its distribution for state $i$,
      \[
      \tilde{y}_{i,2020} \sim P(y_{i,2020} | y_{it}, t < 2020; X_{it'}, t' \leq 2020)
      \]
      i.e. $P(\text{unknown} | \text{data})$. (Details shortly.)
    2. If $\tilde{y}_{i,2020} > 0.5$ Dems “win” $C_i$ electoral college delegates (Reps get 0); otherwise, Dems get 0 (Reps get $C_i$)
  - Calculate total Dem delegates nationally: add simulated winnings from all states:
    \[
    \sum_{i=1}^{51} 1(\tilde{y}_{i,2020} > 0.5)C_i
    \]
  - Calculate QOIs: average, standard deviation, histogram
How to draw simulations of $y_{i,2020}$

1. Choose values of explanatory variables: $X_i,2020$

2. Simulate estimation uncertainty
   - Draw $\eta = \{\hat{\beta}, \hat{\gamma}\}$ from its sampling distribution, $\hat{\eta} \sim N(\hat{\eta}, \hat{V}(\hat{\eta}))$
   - Pull out $\hat{\beta}$ and save
   - Pull out $\hat{\gamma}$, “un-reparameterize” $\hat{\sigma} = e^{\hat{\gamma}}$, and save

3. Compute simulated systematic component: $\hat{\mu}_{it} = X_i,2020 \hat{\beta}$

4. Use stochastic component to simulate fundamental uncertainty: $\hat{y}_{i,2020} \sim N(\hat{\mu}_{i,t}, \hat{\sigma}^2)$

We can now simulate the number of Democratic delegates, in repeated elections, with fundamental and estimation uncertainty represented.
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. Choose values of explanatory variables: $X_c = X_{i,2020}$
2. Simulate estimation uncertainty
   • Draw $\eta = \{\hat{\beta}, \hat{\gamma}\}$ from its sampling distribution, $\hat{\eta} \sim N(\hat{\eta}, \hat{\Sigma})$
   • Pull out $\hat{\beta}$ and save
   • Pull out $\hat{\gamma}$, "un-reparameterize" $\hat{\sigma} = e^{\hat{\gamma}}$, and save
3. Compute simulated systematic component: $\hat{\mu}_{it} = X_{i,2020}\hat{\beta}$
4. Use stochastic component to simulate fundamental uncertainty: $\hat{y}_{i,2020} \sim N(\hat{\mu}_{i,2020}, \hat{\sigma}^2)$

\(\Rightarrow\) We can now simulate the number of Democratic delegates, in repeated elections, with fundamental and estimation uncertainty represented
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. Choose values of explanatory variables: $X_c = X_{i,2020}$
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. Choose values of explanatory variables: $X_c = X_{i,2020}$
2. Simulate estimation uncertainty

$\eta = \{\hat{\beta}, \hat{\gamma}\}$ from its sampling distribution, $\tilde{\eta} \sim N(\hat{\eta}, \hat{V}(\hat{\eta}))$

- Pull out $\hat{\beta}$ and save
- Pull out $\hat{\gamma}$, "un-reparameterize" $\hat{\sigma} = e^{\hat{\gamma}}$, and save

3. Compute simulated systematic component: $\tilde{\mu}_{it} = X_{i,2020}\hat{\beta}$
4. Use stochastic component to simulate fundamental uncertainty: $\tilde{y}_{i,2020} \sim N(\tilde{\mu}_{it}, \tilde{\sigma}^2)$

We can now simulate the number of Democratic delegates, in repeated elections, with fundamental and estimation uncertainty represented.
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. Choose values of explanatory variables: $X_c = X_{i,2020}$
2. Simulate estimation uncertainty
   • Draw $\eta = \{\tilde{\beta}, \tilde{\gamma}\}$ from its sampling distribution,

   $$\tilde{\eta} \sim N(\hat{\eta}, \hat{V}(\hat{\eta}))$$
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. **Choose values of explanatory variables:** $X_c = X_{i,2020}$

2. **Simulate estimation uncertainty**
   - Draw $\eta = \{\tilde{\beta}, \tilde{\gamma}\}$ from its sampling distribution,
     $$\tilde{\eta} \sim N(\hat{\eta}, \hat{V}(\hat{\eta}))$$
   - Pull out $\tilde{\beta}$ and save
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. **Choose values of explanatory variables:** $X_c = X_{i,2020}$
2. **Simulate estimation uncertainty**
   - Draw $\eta = \{\tilde{\beta}, \tilde{\gamma}\}$ from its sampling distribution,
     $$\tilde{\eta} \sim N(\eta, \hat{V}(\eta))$$
   - Pull out $\tilde{\beta}$ and save
   - Pull out $\tilde{\gamma}$, “un-reparameterize” $\tilde{\sigma} = e^{\tilde{\gamma}}$, and save
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. **Choose values of explanatory variables:** $X_c = X_{i,2020}$

2. **Simulate estimation uncertainty**
   - Draw $\eta = \{\tilde{\beta}, \tilde{\gamma}\}$ from its sampling distribution,
     \[ \tilde{\eta} \sim N(\hat{\eta}, \hat{V}(\hat{\eta})) \]
   - Pull out $\tilde{\beta}$ and save
   - Pull out $\tilde{\gamma}$, “un-reparameterize” $\tilde{\sigma} = e^{\tilde{\gamma}}$, and save

3. **Compute simulated systematic component:** $\tilde{\mu}_{it} = X_{i,2020}\tilde{\beta}$
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. **Choose values of explanatory variables:** $X_c = X_{i,2020}$
2. **Simulate estimation uncertainty**
   - Draw $\eta = \{\tilde{\beta}, \tilde{\gamma}\}$ from its sampling distribution,
     
     $\tilde{\eta} \sim N(\hat{\eta}, \hat{V}(\hat{\eta}))$

   - Pull out $\tilde{\beta}$ and save
   - Pull out $\tilde{\gamma}$, “un-reparameterize” $\tilde{\sigma} = e^{\tilde{\gamma}}$, and save
3. **Compute simulated systematic component:** $\tilde{\mu}_{it} = X_{i,2020}\tilde{\beta}$
4. **Use stochastic component to simulate fundamental uncertainty:**

   $\tilde{y}_{i,2020} \sim N(\tilde{\mu}_{i,2020}, \tilde{\sigma}^2)$
How to draw simulations of $y_{i,2020}$
Including fundamental and estimation uncertainty

1. Choose values of explanatory variables: $X_c = X_{i,2020}$
2. Simulate estimation uncertainty
   - Draw $\eta = \{\tilde{\beta}, \tilde{\gamma}\}$ from its sampling distribution,
     $$\tilde{\eta} \sim N(\tilde{\eta}, \tilde{\mathcal{V}}(\tilde{\eta}))$$
   - Pull out $\tilde{\beta}$ and save
   - Pull out $\tilde{\gamma}$, “un-reparameterize” $\tilde{\sigma} = e^{\tilde{\gamma}}$, and save
3. Compute simulated systematic component: $\tilde{\mu}_{it} = X_{i,2020}\tilde{\beta}$
4. Use stochastic component to simulate fundamental uncertainty:
   $$\tilde{y}_{i,2020} \sim N(\tilde{\mu}_{i,2020}, \tilde{\sigma}^2)$$

We can now simulate the number of Democratic delegates, in repeated elections, with fundamental and estimation uncertainty represented.
How to do it with a LS Regression Program

1. Run LS regression of $y_{it}$ on $X_{it}$ and get $\hat{\beta}$ and $V(\hat{\beta})$

2. Draw $\beta$ randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$. Label the random draw $\tilde{\beta}$.

3. Draw $\sigma^2$ from its posterior (or sampling) distribution, $1/\chi^2(\hat{\sigma}^2, N-k)$, labeling it $\tilde{\sigma}^2$.

4. Either:
   - Draw $\epsilon_{it}$ from $N(0, \tilde{\sigma}^2)$, label it $\tilde{\epsilon}_{it}$ and compute:
     $$\tilde{y}_{i,2020} = \tilde{X}_{i,2020} \tilde{\beta} + \tilde{\epsilon}_{it}$$
   - Or, in our preferred notation, draw $\tilde{y}_{i,2020}$ from $N(X_{i,2020} \tilde{\beta}, \tilde{\sigma}^2)$.
How to do it with a LS Regression Program

Useful to connect to the literature. Feel free to ignore

1. Run LS regression of $y_{it}$ on $x_{it}$ and get $\hat{\beta}$ and $V(\hat{\beta})$.

2. Draw $\beta$ randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$.

   Label the random draw $\tilde{\beta}$.

3. Draw $\sigma^2$ from its posterior (or sampling) distribution, $1/\chi^2(\hat{\sigma}^2, N-k)$.

   Label it $\tilde{\sigma}^2$.

4. Either:
   - Draw $\epsilon_{it}$ from $N(0, \tilde{\sigma}^2)$, label it $\tilde{\epsilon}_{it}$ and compute:
     $$\tilde{y}_{i,2020} = \tilde{x}_{i,2020} \tilde{\beta} + \tilde{\epsilon}_{it}$$
   - Or, in our preferred notation, draw $\tilde{y}_{i,2020}$ from $N(x_{i,2020} \hat{\beta}, \tilde{\sigma}^2)$.
How to do it with a LS Regression Program

Useful to connect to the literature. Feel free to ignore

1. Run LS regression of \( y_{it} \) on \( X_{it} \) and get \( \hat{\beta} \) and \( V(\hat{\beta}) \)
How to do it with a LS Regression Program

Useful to connect to the literature. Feel free to ignore

1. Run LS regression of $y_{it}$ on $X_{it}$ and get $\hat{\beta}$ and $V(\hat{\beta})$

2. Draw $\beta$ randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$. Label the random draw $\tilde{\beta}$.
How to do it with a LS Regression Program

Useful to connect to the literature. Feel free to ignore

1. Run LS regression of $y_{it}$ on $X_{it}$ and get $\hat{\beta}$ and $V(\hat{\beta})$

2. Draw $\beta$ randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$. Label the random draw $\tilde{\beta}$.

3. Draw $\sigma^2$ from its posterior (or sampling) distribution, $1/\chi^2(\hat{\sigma}^2, N - k)$, labeling it $\tilde{\sigma}^2$
How to do it with a LS Regression Program

Useful to connect to the literature. Feel free to ignore

1. Run LS regression of $y_{it}$ on $X_{it}$ and get $\hat{\beta}$ and $V(\hat{\beta})$

2. Draw $\beta$ randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$. Label the random draw $\tilde{\beta}$.

3. Draw $\sigma^2$ from its posterior (or sampling) distribution, 

   \[ \frac{1}{\chi^2(\hat{\sigma}^2, N - k)} \]

   labeling it $\tilde{\sigma}^2$

4. Either:
How to do it with a LS Regression Program

Useful to connect to the literature. Feel free to ignore

1. Run LS regression of $y_{it}$ on $X_{it}$ and get $\hat{\beta}$ and $V(\hat{\beta})$

2. Draw $\beta$ randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$. Label the random draw $\tilde{\beta}$.

3. Draw $\sigma^2$ from its posterior (or sampling) distribution, $1/\chi^2(\hat{\sigma}^2, N - k)$, labeling it $\tilde{\sigma}^2$

4. Either:
   - Draw $\epsilon_{it}$ from $N(0, \tilde{\sigma}^2)$, label it $\tilde{\epsilon}_{it}$ and compute:
     $$\tilde{y}_{i,2020} = \tilde{X}_{i,2020}\tilde{\beta} + \tilde{\epsilon}_{it}$$
How to do it with a LS Regression Program
Useful to connect to the literature. Feel free to ignore

1. Run LS regression of $y_{it}$ on $X_{it}$ and get $\hat{\beta}$ and $V(\hat{\beta})$

2. Draw $\beta$ randomly from its posterior distribution (i.e., its sampling distribution), $N(\beta|\hat{\beta}, V(\hat{\beta}))$. Label the random draw $\tilde{\beta}$.

3. Draw $\sigma^2$ from its posterior (or sampling) distribution, $1/\chi^2(\hat{\sigma}^2, N - k)$, labeling it $\tilde{\sigma}^2$.

4. Either:
   - Draw $\epsilon_{it}$ from $N(0, \tilde{\sigma}^2)$, label it $\tilde{\epsilon}_{it}$ and compute:
     $$\tilde{y}_{i,2020} = \tilde{X}_{i,2020}\tilde{\beta} + \tilde{\epsilon}_{it}$$
   - Or, in our preferred notation, draw $\tilde{y}_{i,2020}$ from $N(X_{i,2020}\tilde{\beta}, \tilde{\sigma}^2)$
Forecasting Errors for 1992 (forecasts from early October)

• Predictive distribution of electoral vote proportion:
  • Probability of Dem (Bill Clinton) victory: 0.85
  • Error in Democratic 2-party electoral vote proportion: 0.01
  • Error in Democratic 2-party popular vote proportion: 0.03

Quiz: How big do you expect these errors will be if the model is correct and the election were run again?
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

\[ \text{Probability of Dem (Bill Clinton) victory: 0.85} \]

\[ \text{Error in Democratic 2-party electoral vote proportion: 0.01} \]

\[ \text{Error in Democratic 2-party popular vote proportion: 0.03} \]

Quiz: How big do you expect these errors will be if the model is correct and the election were run again?
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

- Probability of Dem (Bill Clinton) victory:
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

- Probability of Dem (Bill Clinton) victory: 0.85

![Graph showing posterior distribution of 1992 presidential election outcome]
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

  ![Posterior Distribution of 1992 Presidential Election Outcome]

  - Probability of Dem (Bill Clinton) victory: 0.85
  - Error in Democratic 2-party electoral vote proportion:

Simulation from Likelihood Models
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

- Probability of Dem (Bill Clinton) victory: 0.85
- Error in Democratic 2-party electoral vote proportion: 0.01

![Posterior Distribution of 1992 Presidential Election Outcome](image-url)
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

![Graph showing posterior distribution of 1992 presidential election outcome.]

- Probability of Dem (Bill Clinton) victory: 0.85
- Error in Democratic 2-party electoral vote proportion: 0.01
- Error in Democratic 2-party popular vote proportion:
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

![Posterior Distribution of 1992 Presidential Election Outcome](image)

- Probability of Dem (Bill Clinton) victory: 0.85
- Error in Democratic 2-party electoral vote proportion: 0.01
- Error in Democratic 2-party popular vote proportion: 0.03
Forecasting Errors for 1992 (forecasts from early October)

- Predictive distribution of electoral vote proportion:

  - Probability of Dem (Bill Clinton) victory: 0.85
  - Error in Democratic 2-party electoral vote proportion: 0.01
  - Error in Democratic 2-party popular vote proportion: 0.03
  - Quiz: How big do you expect these errors will be if the model is correct and the election were run again?
The Impossibility of Inference Without Assumptions

Three Theories of Inference: Overview

Likelihood: Example, Derivation, Properties

Uncertainty in Likelihood Inference

Simulation from Likelihood Models

Extending the Linear Model with a Variance Function
A Gaussian Variance Function Model

1. \( Y_i \sim N(\mu_i, \sigma^2_i) \)

2. \( \mu_i = X_i \beta \), with covariates \( X_i \)

3. \( \sigma^2_i = \exp(z_i \gamma) \), with covariates \( z_i \) possibly overlapping \( X_i \)

4. \( Y_i \) and \( Y_i' \) are independent \( \forall i \neq i' \), given \( X \) and \( Z \).

The Log-Likelihood Derivation

\[
\ln L(\beta, \sigma^2 | y) = -\frac{1}{2} n \sum_{i=1}^{n} \left[ \ln \sigma^2 + (y_i - \mu_i)^2 / \sigma^2 \right] = -\frac{1}{2} n \sum_{i=1}^{n} \left[ z_i \gamma + (y_i - X_i \beta)^2 \exp(z_i \gamma) \right]
\]

Any questions?
A Gaussian Variance Function Model

The Model

\[ Y_i \sim N(y_i | \mu_i, \sigma_i^2) \]

\[ \mu_i = x_i \beta, \text{ with covariates } x_i \]

\[ \sigma_i^2 = \exp(z_i \gamma), \text{ with covariates } z_i \text{ possibly overlapping } x_i \]

\[ Y_i \text{ and } Y_i' \text{ are independent } \forall i \neq i' \text{, given } X \text{ and } Z. \]
A Gaussian Variance Function Model

The Model

1. \( Y_i \sim N(y_i | \mu_i, \sigma_i^2) \)
A Gaussian Variance Function Model

The Model

1. \( Y_i \sim N(y_i|\mu_i, \sigma_i^2) \)
2. \( \mu_i = x_i\beta \), with covariates \( x_i \)
A Gaussian Variance Function Model

The Model

1. $Y_i \sim N(y_i | \mu_i, \sigma_i^2)$
2. $\mu_i = x_i \beta$, with covariates $x_i$
3. $\sigma_i^2 = \exp(z_i \gamma)$, with covariates $z_i$ possibly overlapping $x_i$
A Gaussian Variance Function Model

The Model

1. $Y_i \sim N(y_i | \mu_i, \sigma_i^2)$
2. $\mu_i = x_i \beta$, with covariates $x_i$
3. $\sigma_i^2 = \exp(z_i \gamma)$, with covariates $z_i$ possibly overlapping $x_i$
4. $Y_i$ and $Y_{i'}$ are independent $\forall i \neq i'$, given $X$ and $Z$. 

Any questions?
A Gaussian Variance Function Model

The Model

1. \( Y_i \sim N(y_i|\mu_i, \sigma_i^2) \)
2. \( \mu_i = x_i \beta \), with covariates \( x_i \)
3. \( \sigma_i^2 = \exp(z_i \gamma) \), with covariates \( z_i \) possibly overlapping \( x_i \)
4. \( Y_i \) and \( Y_{i'} \) are independent \( \forall i \neq i' \), given \( X \) and \( Z \).

The Log-Likelihood Derivation
A Gaussian Variance Function Model

The Model

1. \( Y_i \sim N(y_i|\mu_i, \sigma_i^2) \)
2. \( \mu_i = x_i\beta \), with covariates \( x_i \)
3. \( \sigma_i^2 = \exp(z_i\gamma) \), with covariates \( z_i \) possibly overlapping \( x_i \)
4. \( Y_i \) and \( Y_i' \) are independent \( \forall i \neq i' \), given \( X \) and \( Z \).

The Log-Likelihood Derivation

\[
\ln L(\beta, \sigma^2|y) = -\frac{1}{2} \sum_{i=1}^{n} \left[ \ln \sigma^2 + \frac{(y_i - \mu_i)^2}{\sigma^2} \right]
\]
A Gaussian Variance Function Model

The Model

1. $Y_i \sim N(y_i|\mu_i, \sigma_i^2)$
2. $\mu_i = x_i \beta$, with covariates $x_i$
3. $\sigma_i^2 = \exp(z_i \gamma)$, with covariates $z_i$ possibly overlapping $x_i$
4. $Y_i$ and $Y_{i'}$ are independent $\forall i \neq i'$, given $X$ and $Z$.

The Log-Likelihood Derivation

$$\ln L(\beta, \sigma^2|y) = -\frac{1}{2} \sum_{i=1}^{n} \left[ \ln \sigma^2 + \frac{(y_i - \mu_i)^2}{\sigma^2} \right]$$
$$= -\frac{1}{2} \sum_{i=1}^{n} \left[ z_i \gamma + \frac{(y_i - X_i \beta)^2}{\exp(z_i \gamma)} \right]$$
A Gaussian Variance Function Model

The Model

1. \( Y_i \sim N(y_i|\mu_i, \sigma_i^2) \)
2. \( \mu_i = x_i \beta \), with covariates \( x_i \)
3. \( \sigma_i^2 = \exp(z_i \gamma) \), with covariates \( z_i \) possibly overlapping \( x_i \)
4. \( Y_i \) and \( Y_{i'} \) are independent \( \forall \ i \neq i' \), given \( X \) and \( Z \).

The Log-Likelihood Derivation

\[
\ln L(\beta, \sigma^2|y) = -\frac{1}{2} \sum_{i=1}^{n} \left[ \ln \sigma^2 + \frac{(y_i - \mu_i)^2}{\sigma^2} \right] \\
= -\frac{1}{2} \sum_{i=1}^{n} \left[ z_i \gamma + \frac{(y_i - X_i \beta)^2}{\exp(z_i \gamma)} \right]
\]

Any questions?